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J.M. Jastrzebski, Instytut Matematyki, Uniwersytet Gdanski, ul. Wita Stwosza 57, Gdansk, Poland.

J.M. Jedrzejewski, Instytut Matematyki WSP, ul Chodkiewicza 30, Bydgoszcz, Poland.

T. Natkaniec, Instytut Matematyki WSP, ul Chodkiewicza 30, Bydgoszcz, Poland.

Points of Almost Continuity of Real Functions

Many authors considered the local property of Darboux or local connectivity of a real function and the sets of those points at which a real function of a real variable has the local Darboux property ([1]) or local connectivity property ([2]). The functions that are almost continuous are connected and hence have the Darboux property. There arises an interesting question whether it is possible to define a local property of almost continuity at a point and moreover to make sure that this property characterize the global almost continuity of a function. We give such a characterization in this article.

Throughout the article we shall make use of the following notions and notations.

Consider the following topological properties that a function f from one topological space X to another one Y may have:

$\mathcal{C}$: f is continuous,

$\mathcal{D}$: f is a Darboux function iff for every connected subset C of X, the set $f(C)$ is a connected subset of Y.

$\mathcal{C}on$: f is a connectivity function iff for every connected subset C of X the set $f|C$ (the graph of this function over C) is a connected subset of $X \times Y$. In the sequel, we shall make no difference between a function and its graph.

$\mathcal{AC}$: f is almost continuous function (in the sense of Stallings [4]) iff for every open set $G \subseteq X \times Y$ containing f there exists a continuous function $g : X \to Y$ such that $g \subseteq G$.

The functions we consider are real functions defined on an interval (open or closed). Recall that for those functions the following implications hold:

$$\mathcal{C} \Rightarrow \mathcal{AC} \Rightarrow \mathcal{C}on \Rightarrow \mathcal{D} \qquad ([4])$$

By $L(f,x)$, $L^+(f,x)$, $L^-(f,x)$ we shall denote: the set of all limit points, the set of all right-hand sided limit points, the set of all left-hand sided limit points of the function f at the point x, respectively.

Definition 1. A function $f : (a,b) \to \mathbf{R}$ is said to be almost continuous at a point $x \in (a,b)$ from the right-hand side iff

(1) $f(x) \in L^+(f,x)$,

(2) there is a positive ε such that for any arbitrary neighborhood G of $f \mid [x,\infty)$, arbitrary $y \in (\liminf_{t\to x^+} f(t), \limsup_{t\to x^+} f(t))$, arbitrary neighborhood U of the point (x,y) and arbitrary $t \in (x, x+\varepsilon)$ there exists a continuous function $g : [x, x+\varepsilon] \to \mathbf{R}$ such that $g \subseteq G \cup U$, $g(x) = y$ and $g(t) = f(t)$.

In the analogous manner we define a function that is almost continuous at a point from the left-hand side. A function is called almost continuous at a point iff it is almost continuous at that point from both sides. For a function $f : [a,b] \to \mathbf{R}$ we can say that it is almost continuous at the points a and b if it is so from the right-hand side at a and from the left-hand side at b.

One can easily observe that if a function is continuous at a point from any side then it is almost continuous at that point from the same side. Moreover:

Property 1. If a function $f : (a,b) \to \mathbf{R}$ is right-hand sided almost continuous at a point from (a,b) then it is connected at that point from the same side.

Proof. Suppose that f is right-hand sided almost continuous at a point x_0 but is not connected at it from the right-hand side. Then there exists a continuum M contained in $[x_0,\infty) \times \mathbf{R}$ such that

(i) $\mathrm{proj}_y M \subseteq (\liminf_{x\to x_0^+} f(x), \limsup_{x\to x_0^+} f(x))$,

(ii) $\mathrm{proj}_x M$ is nondegenerate,

(iii) $M \cap f = \emptyset$ (see [2]).

Then $\mathbf{R}^2 \setminus M$ is a neighborhood of f. There exists a point

$$y \in (\liminf_{x\to x_0^+} f(x), \limsup_{x\to x_0^+} f(x)) \setminus \mathrm{proj}_y(M \cap \mathcal{P}(x_0)),$$

where $\mathcal{P}(x) = \{x\} \times \mathbf{R}$.

Let $[m_1, m_2] = \mathrm{proj}_y(M \cap \mathcal{P}(x_0))$ and assume that $y < m_1$. Then for every $n \in \mathbf{N}$ there is x_n such that $x_0 < x_n < x_0 + 1/n$ and $f(x_n) > m_2$. There is an open neighborhood U of the point (x_0, y) disjoint with M, so $\mathbf{R}^2 \setminus M$ is a neighborhood of f and each continuous function $g : [x_0, x_n] \to \mathbf{R}$, for which $g(x_0) = y$, $g(x_n) = f(x_n)$ has common points with M, which contradicts to the almost continuity of f at the point x_0 from the right-hand side.

As a corollary of Property 1 we get that a function is Darboux at a point if it is almost continuous at that point.

Now we shall prove that the local and global properties of almost continuity are compatible.

Theorem 1. If $f : (a, b) \to \mathbf{R}$ is almost continuous, then it is almost continuous at every point of the interval (a, b).

Proof. If f is continuous at a point x_0 from the right-hand side then, of course, it is almost continuous at that point from the right-hand side. If not, then $L^+(f, x_0)$ is a nondegenerate interval. Let y be an arbitrary point from the interval

$$(\inf L^+(f, x_0), \sup L^+(f, x_0))$$

and G be an arbitrary neighborhood of $f \mid [x_0, b)$, U - arbitrary neighborhood of (x_0, y) and t - a point from (x_0, b). There is a point of f contained in U, let it be $(x_1, f(x_1))$, and we can take that $x_1 > x_0$. There is a function $g_1 : [x_0, x_1] \to \mathbf{R}$ such that

(a) g_1 is continuous,

(b) $g_1 \subseteq U$

(c) $g_1(x_0) = y$ and $g_1(x_1) = f(x_1)$.

The function f is almost continuous in (a, b), then for its neighborhood G and the point t from $[x_1, b)$ there is a function $g_2 : [x_1, b) \to \mathbf{R}$ such that

(a′) g_2 is continuous,

(b′) $g_2 \subseteq G$,

(c′) $g_2(x_1) = f(x_1)$ and $g_2(t) = f(t)$.

In this way the function $g : [x_0, b) \to \mathbf{R}$ given by

$$g(x) = \begin{cases} g_1(x) & \text{for} \quad x \in [x_0, x_1], \\ g_2(x) & \text{for} \quad x \in (x_1, b) \end{cases}$$

is continuous and $g \subseteq G \cup U$. This proves that f is almost continuous at x_0 from the right-hand side. Similarly one can prove that f is also almost continuous at x_0 from the left-hand side.

Theorem 2. If a function $f : [a,b] \to \mathbf{R}$ is almost continuous at every point of the interval $[a,b]$, then f is almost continuous in $[a,b]$.

Proof. Let G be any neighborhood of f. If f is continuous at x_0, then there is a square U with the centre at $(x_0, f(x_0))$ contained in G. Consider three possibilities:

- f is continuous at x_0,

- f is not continuous at x_0 but

$$f(x_0) \in (\inf L^+(f,x_0), \sup L^+(f,x_0)) \cap (\inf L^-(f,x_0), \sup L^-(f,x_0)),$$

- x_0 is a point of discontinuity of f and either

$$f(x_0) = \sup L^-(f,x_0) \text{ or } f(x_0) = \inf L^-(f,x_0),$$

$$\text{or } f(x_0) = \sup L^+(f,x_0), \text{ or } f(x_0) = \inf L^+(f,x_0).$$

In each possibility, for $x \in [a,b]$ there is $\varepsilon_x > 0$ such that for every $t_1, t_2 \in (x - \varepsilon_x, x + \varepsilon_x)$ for which $t_1 < x < t_2$ there exists a continuous function $g_{x,t_1,t_2} : [x - \varepsilon_x, x + \varepsilon_x] \to \mathbf{R}$ fulfilling all the conditions:

(α) $g_{x_0,t_1,t_2}(x_0) = f(x_0)$,

(β) $g_{x_0,t_1,t_2}(t_i) = f(t_i)$, for $i = 1,2$,

(γ) $g_{x_0,t_1,t_2} \subseteq G$.

The family $\{(x - \varepsilon_x, x + \varepsilon_x) : x \in [a,b]\}$ is a cover of the interval $[a,b]$, so there exists a finite sequence of points and a sequence of continuous functions defined on those intervals, which joined together form a continuous function contained in G.

It is obvious that if $f : (a, a_0] \to \mathbf{R}$ has the property that $f \mid [a_n, a_0]$ is almost continuous, where $a_n \in (a, a_0)$ and $a_n \to a$, then f is almost continuous in $(a, a_0]$.

Now we are able to state the following:

Theorem 2′. If $f : (a,b) \to \mathbf{R}$ is almost continuous at every point of the interval (a,b), then it is almost continuous in (a,b).

For the points of almost continuity of a function we can prove the analogue of the theorem on asymmetry.

Theorem 3. The set of all points of the interval (a, b), at which a function $f : (a, b) \to \mathbf{R}$ is almost continuous from exactly one side is at most countable.

Proof. Let A be the set of asymmetry of almost continuity of a function f. Let us denote by B the set of all points, at which f is almost continuous from the right-hand side and is not almost continuous from the left-hand side. By C we denote the set $A \setminus B$. Let

$$D_1 = \{x \in (a, b) : L^+(f, x) \neq L^-(f, x)\},$$
$$D_2 = \{x \in (a, b) : f(x) \notin L^+(f, x) \cap L^-(f, x)\},$$
$$E = B \setminus (D_1 \cup D_2).$$

The sets D_1, D_2 are countable ([5]). We shall show that the set E is also countable.

Let E_n be the set of all points $x_0 \in E$ such that the diameter of the set $L(f, x_0)$ is greater than or equal to $1/n$ and such that for: every neighborhood G of $f \mid [x_0, x_0 + 1/n]$, every $t \in (x_0, x_0 + 1/n]$ and each y from the interval $(\inf L^+(f, x_0), \sup L^+(f, x_0))$ and every neighborhood U of (x_0, y) there exists a continuous function $g : [x_0, x_0 + 1/n] \to \mathbf{R}$ such that $g \subseteq G \cup U$, $g(x_0) = y$ and $g(t) = f(t)$. One can prove that each of the sets E_n is countable, so is E. Similarly, C is countable and, of course, so is A.

Theorem 4. The set of all points of almost continuity of an arbitrary function is of the type G_δ.

Proof. Let $f : (a, b) \to \mathbf{R}$ be an arbitrary function. By $\mathcal{A}^+(f)$ ($\mathcal{A}^-(f)$) we shall denote the set of all points of (a, b) at which f is almost continuous from the right-hand side (left-hand side), and $\mathcal{A}(f) = \mathcal{A}^+(f) \cap \mathcal{A}^-(f)$. Let A_n be the set of those points of $\mathcal{A}(f)$ for which the ε from the definition 1 is greater than $1/n$. We shall show that

$$A_n \subseteq \operatorname{Int}(A_n \cup \mathcal{C}(f)),$$

where $\mathcal{C}(f)$ denotes the set of all points of continuity of the function f. Let $x_0 \in A_n$ and $\delta = \varepsilon - 1/n$. We shall prove that

$$(x_0 - \delta, x_0 + \delta) \subseteq A_n \cup \mathcal{C}(f).$$

If $x \in (x_0 - \delta, x_0 + \delta) \setminus \mathcal{C}(x)$ and, for example, $x_0 < x$, G is an arbitrary neighborhood of f, $y \in (\inf L(f, x), \sup L(f, x))$ and U is a square neighborhood of (x, y), then there are t', t'' such that

$$t' < x < t'', \ (t', f(t')) \in U \ \text{and} \ (t'', f(t'')) \in U.$$

Assume that $U = (x - \eta, x + \eta) \times (y - \eta, y + \eta)$, and let $t \in (x + \eta, x_0 + \varepsilon)$. Then

$$G' = (G \cup U) \setminus (\{t', t''\} \times ((-\infty, y - \eta] \cup [y + \eta, \infty)))$$

is a neighborhood of $f \mid [x_0, \infty)$ and there exists a continuous function g such that $g(x_0) = f(x_0)$ and $g(t) = f(t)$. The function g meets the square U and then there is a continuous function $h : [x, x_0 + \varepsilon) \to \mathbf{R}$ such that $h(x) = y$, $h(t) = f(t)$ and $h \subseteq G' \subseteq G \cup U$. Similarly, for $t \in (x - \varepsilon, x)$ there exists a continuous function $h' : [x - \varepsilon, x] \to \mathbf{R}$ such that $h' \subseteq G \cup U$, $h'(x) = y$ and $h'(t) = f(t)$. Thus if $x_0 \in A_n$ then $x_0 \in \mathrm{Int}(A_n \cup \mathcal{C}(f))$. Since $\mathcal{A}(f) = \bigcup_{n \in \mathbf{N}} A_n \cup \mathcal{C}(f)$ and

$$\bigcup_{n \in \mathbf{N}} A_n \cup \mathcal{C}(f) = \bigcup_{n \in \mathbf{N}} \mathrm{Int}(A_n \cup \mathcal{C}(f)) \cup \mathcal{C}(f),$$

then $\mathcal{A}(f)$ is of the type G_δ.

In the end of the article it is worth mentioning that Theorem 4 gives an exact characterization of the set of all points of almost continuity of a function. Indeed, J.S. Lipiński in [3] has proved that for a given set A of the type G_δ there is a function f for which $\mathcal{C}(f) = A$ and $\mathbf{R} \setminus A$ is the set of all points at which f does not have the Darboux property. Such a function fulfils also our requirements, i.e. $\mathcal{A}(f) = A$.

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