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A Note on Continuity Points of Functions

§1.

Using the fact that $\mathbf{R}$ is locally connected and locally compact, it can be shown that if $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a separately continuous function with a closed graph, then f is continuous. Instead of proving this result, we will consider the question of how it might be generalized. Specifically, what conditions on spaces X, Y, and Z are necessary and sufficient to guarantee that a separately continuous function $f : X \times Y \to Z$ with a closed graph is continuous? We give two examples that show some limitations.

Example 1: Let $X = Y = [0,1] - \{\frac{1}{n} : n \in \mathbf{N}\}$ with the usual topology. Notice that X is not locally connected since 0 does not have a connected neighborhood. Define $f : X \times Y \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} n, & \text{if } x, y \in (\frac{1}{n+1}, \frac{1}{n}) \text{ for some } n \in \mathbf{N} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f is separately continuous, has a closed graph, but is not continuous at $(0,0)$.

As we shall show, if X or Y is locally connected, then a function with the properties mentioned above will be continuous. In fact, we may replace the codomain $\mathbf{R}$ by any locally compact space Z. But what if Z is not locally compact?

Example 2: Let $I = [0,1]$ and let Z be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let ϕ be defined by

$$\phi(x,y) = \begin{cases} 1 - x - y^2, & \text{if } x^2 + y^2 \leq 1 \\ 0, & \text{if } x^2 + y^2 > 1 \end{cases}$$

and let

$$\phi_n(x,y) = \phi(2n(n+1)x - (2n+1), 2n(n+1)y - (2n+1)),$$

for each $n \in \mathbb{N}$. Each function ϕ_n is 1 at the center $\left(\frac{2n+1}{2n(n+1)}, \frac{2n+1}{2n(n+1)}\right)$ of the circle inscribed in the square

$$\left[\frac{1}{n+1}, \frac{1}{n}\right] \times \left[\frac{1}{n+1}, \frac{1}{n}\right]$$

and vanishes outside of this circle. Define $f : I \times I \to X$ by $f(x,y) = \sum_{n=1}^{\infty} \phi_n(x,y)e_n$. Then on each square $(\frac{1}{n+1}, \frac{1}{n}) \times (\frac{1}{n+1}, \frac{1}{n})$ we have $f(x,y) = \phi_n(x,y)e_n$ and outside these squares f vanishes. It is easy to see that at each $(x,y) \neq (0,0)$ f is continuous, and since $f(0,x) = f(x,0) = 0$ for each $x \in I$, f is separately continuous at $(0,0)$. In addition to this, f has a closed graph. However, f is not continuous at $(0,0)$ since

$$||f\left(\frac{2n+1}{2n(n+1)}, \frac{2n+1}{2n(n+1)}\right) - f(0,0)|| = ||e_n|| = 1$$

for every $n \in \mathbb{N}$.

§2.

As we have seen in the first section, we cannot guarantee that a separately continuous function $f : X \times Y \to Z$ with a closed graph will be continuous if neither X nor Y is locally connected or if Z is not locally compact. However, we have the following theorem.

Theorem 1. *Let X and Y be topological spaces with Y locally connected. Let Z be locally compact and suppose that $f : X \times Y \to Z$ has continuous y-sections and connected x-sections. If f has a closed graph, then f is continuous.*

Proof: Let $(a,b) \in X \times Y$ and suppose f is not continuous at (a,b). Then there is a neighborhood W of $f(a,b)$ such that, for any neighborhood N of (a,b), $f(N) \not\subset W$. Since Z is locally compact, we may assume that $\bar{W}$ is compact. Let $\mathcal{D}$ be the set of all neighborhoods $U \times V$ of (a,b) such that $f(U,b) \subset W$ and V is connected. Because of the continuity of the y-sections of f and the local connectedness of Y, the set $\mathcal{D}$ is a neighborhood basis at (a,b). Also $\mathcal{D}$ can be directed by containment (that is, $\alpha \leq \beta$ if $\alpha \supset \beta$). Let $\alpha = U \times V$ be an element of $\mathcal{D}$. Since $f(U \times V) \not\subset W$, there is a point $(x,y) \in U \times V$ such that $f(x,y) \notin W$, and since $f(U,b) \subset W$, $f(x,b) \in W$. The set $f(x,V)$ is connected because x-sections of f are connected. Hence there is a point $(x_\alpha, y_\alpha) \in U \times V$ such that $f(x_\alpha, y_\alpha) \in \bar{W} - W$. Now $(f(x_\alpha, y_\alpha) : \alpha \in \mathcal{D})$ is a net in the compact set $\bar{W} - W$. Hence it contains a convergent subnet $(f(x_{n(\alpha)}, y_{n(\alpha)}) : \alpha \in \mathcal{D}')$, which converges to some point $c \in \bar{W} - W$. Because $\mathcal{D}$ is a neighborhood basis at (a,b), the net

$$((x_{n(\alpha)}, y_{n(\alpha)}, f(x_{n(\alpha)})) : \alpha \in \mathcal{D}')$$

converges to (a, b, c), which implies that $c = f(a, b)$ since f has a closed graph. This is impossible since $f(a, b) \in W$ and $c \in \bar{W} - W$. Therefore f is continuous.

Remark 1: As an immediate consequence of Theorem 1 we have that if X is locally connected, Y is locally compact, and $f : X \to Y$ is a connected mapping with a closed graph, then f is continuous. This can be easily seen by applying the theorem to the function $\tilde{f} : \{0\} \times X \to Y$ defined by $\tilde{f}(0, x) = f(x)$.

§**3.**

The second part of this paper will deal with the problem of finding the weakest assumptions on spaces X, Y and Z and the sections f_x, f^y of functions $f : X \times Y \to Z$ such that f has at least one point of (joint) continuity.

One source of the results of this nature is the Baire-Lebesgue-Kuratowski-Montgomery theorem which says that if X and Y are metric and if $f : X \times Y \to R$ is continuous in x and is of class α in y, then f is of class $(\alpha + 1)$. Now, if $\alpha = 0$ and $X \times Y$ is Baire, then the set $C(f)$ is a dense G_δ subset of $X \times Y$ by Baire's Theorem, f being of 1^{st} class (see [P3] for further discussion on this topic).

Recently, G. Debs [De] has shown that if X is a special α-favorable pace (thus Baire), Y is first countable, $X \times Y$ is Baire, and $f : X \times Y \to M$ (M-metric) is such that all of its x-sections f_x are continuous and all of its y-sections f^y are, what he calls, of "first class", then the set $C(f)$ is dense in $X \times Y$. (This result was unknown even in the case when $X = Y = M = [0, 1]$.)

Remark 2: The first-named author has obtained very similar results (see [P1] and [P2]) using an actually larger class of spaces X (the entire class of Baire spaces) and the somewhat unrelated class of functions f whose y-sections f^y are quasi-continuous[1] (instead of "first class") together with a strengthened form of the conclusion, namely:

If X is Baire, Y is first countable and Z is metric and if a function $f : X \times Y \to Z$ has all its x-sections f_x continuous and has all of its y-sections f^y quasi-continuous, then for all $y \in Y$, the set $C(f)$ is a dense G_δ subset in $X \times \{y\}$.

Following N.F.G. Martin [Ma], a function $f : X \to Y$ is called *quasi-continuous* if for every $x \in X$, for every open set U containing x, and for every open set V containing $f(x)$, there is an open nonempty set $U' \subset U$ such that $f(U') \subset V$.

A class of functions that is closely related to functions of first class of Baire is the class of pointwise discontinuous functions (see [Ku]). $f : X \to Y$ is *pointwise discontinuous* (or, shortly: PWD) if the set $C(f)$ of points of continuity is dense

[1] This class of functions has been defined by V. Volterra in R. Baire's paper [Ba] p. 75.

in the domain of f[2].

R. Baire showed the following result:

Theorem. *(R. Baire) If $f : \mathbf{R} \to \mathbf{R}$ is of the first class of Baire, it is PWD.*

The converse to this theorem is not true (!) — see J.C. Oxtoby [O2].

For "nice" spaces, say $X = Y = \mathbf{R}$, we have the following diagram (where "$\longrightarrow$" denotes the inclusion):

$$\begin{tikzcd}
& \text{QUASI-CONTINUITY} \arrow[dr] & \\
\text{CONTINUITY} \arrow[ur] \arrow[dr] & \not\downarrow\ \not\uparrow & \text{PWD} \\
& \text{1ST CLASS} \arrow[ur] &
\end{tikzcd}$$

The survey paper [Ne] contains proofs of the implications pertaining to quasi-continuity in the above diagram.

In this section we strengthen the result of G. Debs and the just mentioned result by the first-named author.

The following Lemma clearly follows from Baire Category Theorem.

Lemma: ([DŠ]), Theorem 1.1 and 1.2, p. 220).
Let X be a Baire space and let M be metric. Then $f : X \to M$ is PWD iff f satisfies the following condition:

> (*) for every $x \in X$, for every $\varepsilon > 0$, and for every neighborhood $U(x)$ of x there exists an open, nonempty set $U, U \subset U(x)$, such that $d(f(z), f(y)) < \varepsilon$ for any two points $z, y \in U$.

A pseudo-base, or simply a π-base, (see [O1]) for a space $(X, \mathcal{T})$ is a subset $\mathcal{P}$ of $\mathcal{T}$ such that every nonempty element U of $\mathcal{T}$ contains a nonempty element G of $\mathcal{P}$.

Theorem 2. *Let X be Baire and Y be locally of π-countable type (i.e., each open nonempty subset of Y contains an open nonempty subset having a countable π-base) such that $X \times Y$ is Baire. Further let (M, d) be a metric space. Let $f : X \times Y \to M$ be a function such that all of its x-sections f_x are PWD and all of its y-sections f^y are continuous. Then $C(f)$ is a dense G_δ subset of $X \times Y$.*

Proof: Given an arbitrary $(x_0, y_0) \in X \times Y$, let U and V be open neighborhoods of x_0 and y_0, respectively. Fix $\varepsilon > 0$. Further assume V contains an open subset having a countable π-base $\{G_n\}$.

[2]This class of functions was defined by H. Hankel in 1870.

Define the set A_n by
$A_n = \{x \in U :$ there are open $V_x \subset V$ and $G_n \subset V_x$ such that for each

$$y_1, y_2 \in V_x \text{ we have } d(f(x,y_1), f(x,y_2)) < \varepsilon/8\}.$$

For $x \in U$, f_x (being PWD) satisfies $(*)$, so there is a nonempty open set $V_x \subset V$ such that for each $y_1, y_2 \in V_x$ we have $d(f(x,y_1), f(x,y_2)) < \varepsilon/8$. Since $\{G_n\}$ is a π-base for an open nonempty subset of V, there is an index n such that $G_n \subset V_x$, and it follows that $U \subset \cup_{n\in N} A_n$. Since by definition $U \supset \cup_{n\in N} A_n$, $U = \cup_{n\in N} A_n$.

X being Baire, U is of second category. So, there is an index $n \in \mathbf{N}$ and a nonempty open set $U' \subset U$ such that $A_n \cap U'$ is dense in U'. Let $(p,q) \in U' \times G_n$. Since f^q is continuous, there is an open nonempty subset $U'' \subset U'$ such that for each $x_1, x_2 \in U''$ we have $d(f(x_1,q), f(x_2,q)) < \frac{\varepsilon}{8}$.

Now consider the set

$$S = (U'' \times \{q\}) \cup ((A_n \cap U'') \times G_n).$$

It is easy to see that $int\bar{S} \neq \emptyset$.

Now, take $(x,y) \in U'' \times G_n$ and $(u,v) \in S$. By continuity of f^y, there is an open set $U_y \subset U''$ such that for each $x_1 \in U_y$ we have $d(f(x,y), f(x_1,y)) < \frac{\varepsilon}{8}$. Since $A_n \cap U''$ is dense in U'' there is $x^* \in U_y \cap U'' \cap A_n$. This gives $(x^*, y) \in S$.

Thus we get the following estimate:

$$d(f(x,y), f(u,v)) \leq d(f(x^*,y), f(x,y)) + d(f(x^*,y), f(x^*,q)) +$$
$$+ d(f(x^*,q), f(u,q)) + d(f(u,q), f(u,v)) < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{2}.$$

This way for each $(x^0,y^0), (x,y) \in U'' \times G_n$ we get

$$d(f(x^0,y^0), f(x,y)) < \varepsilon.$$

Now, since $U'' \times G_n$ is an open, nonempty subset of $U \times V$, we have proved that f satisfies $(*)$ at (x_0, y_0) and hence, by the Lemma, $C(f)$ is a dense G_δ, M being metric and $X \times Y$ being Baire.

We shall now exhibit an example showing that the assumption that the y-sections are continuous in the Theorem is real; that is, it can not be weakened to the one that the y-sections are assumed to be (only) PWD.

Example 3: Let $I = [0,1]$ and let $\mathbf{R}$ be the set of reals. Put $D_n = \{(x,y) : x = \frac{k}{2^n}, y = \frac{p}{2^n}$, where k and p are all odd numbers between 0 and $2^n\}$. Let $D = \cup_{n=1}^{\infty} D_n$. It is easy to see that $\bar{D} = I^2$. Now, let us define $f : I^2 \to \mathbf{R}$ by:

$f(x,y) = 1$ for $(x,y) \in D$ and $f(x,y) = 0$ if $(x,y) \notin D$. The function f is not PWD as a function of two variables, however each section f_x and f^y is PWD — every such section has finitely many "points of jump" of f.

Remark 3: Example 3 can be generalized to the following result, (see [P4], pp. 77, 78):
Let X and Y be dense-in-themselves, separable spaces and let Z be a Hausdorff space containing at least two points. Then there is a function $f : X \times Y \to Z$ such that all the x-sections f_x and all the y-sections f^y are PWD, while f is not PWD.

Remark 4: The result mentioned in Remark 2 can be further generalized; the assumption "Y is first countable" can be weakened to "Y contains a dense subspace of points of first countability".

Remark 5: Both Debs's Theorem and our Theorem 4 are partial (positive) answers to a spectacular problem of M. Talagrand [Ta]: Let X be Baire, Y be compact and let $f : X \times Y \to \mathbf{R}$ be separately continuous. Is $C(f) \neq \emptyset$?

References

[Ba] R. Baire, Sur les fonctions des variables reélles, Ann. Math. Pura Appl. 3 (1899), 1-122.

[Ch] J.P.R. Christensen, Joint continuity of separately continuous functions, Proc. Amer. Math. Soc. 82 (1981), 455-461, MR82 #54012.

[De] G. Debs, Fonctions séparément continues et de première classe sur espace produit, Math. Scand. 59 (1986), 122-130, MR88c #54014.

[DŠ] J. Doboš, T. Šalát, Cliquish functions, Riemann integrable functions and quasi-uniform convergence, Acta Math. Univ. Comen., XL-XLI (1982), 219-223, MR84a #54003.

[Ku] K. Kuratowski, Topology, vol. I, Warsazawa, 1966.

[Ma] N.F.G. Martin, Quasi-continuous functions on product spaces, Duke J. Math. 28 (1961), 39-44.

[Ne] T. Neubrunn, Quasi-continuity, Real Analysis Exchange 14 (1988-89), 259-306.

[O1] J.C. Oxtoby, Cartesian products of Baire spaces, Fund. Math 49 (1961), 157-166.

[O2] ________, Measure and category, Springer, New York.

[P1] Z. Piotrowski, Quasi-continuity and product spaces, Proc. Intern. Conf. Geom. Topology PWN, Warsaw (1980), 349-352.

[P2] ________, Continuity points in $\{x\} \times Y$, Bull. Soc. Math. France 108 (1980), 113-115.

[P3] ________, Separate and joint continuity, Real Analysis Exchange, 11 (1985-86), 293-322.

[P4] ________, Some remarks on almost continuous functions, Math. Slovaca, 39 (1989), 75-80.

[Ta] M. Talagrand, Espaces de Baire et espaces de Namioka, Math. Ann. 270 (1985), 159-164.

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