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ESSAYS ON THE ORLICZ—PETTIS THEOREM, I (THE TWO THEOREMS)

To the memory of W. Orlicz

The purpose of this article that we plan to publish in a series of notes, is a presentation of the results that concentrate around (what is now usually called) "the Orlicz–Pettis Theorem", in their historical perspective.

This theorem is believed to be one of the great theorems of the classical period of functional analysis. In 1979 a conference was held in Chapel Hill, North Carolina, in memory of B. J. Pettis, and Kalton gave then a short survey [1980] of the theorem. As he put it in his talk, the theorem "during the course of its evolution has evolved almost beyond recognition, and the techniques developed for its study have themselves helped to illuminate a number of ideas in functional analysis".

Our ambition here is to give not only the full history but also a thorough discussion of the theorem. These are the reasons for doing so:

1) The Orlicz-Pettis Theorem has been a source of misunderstandings since its very beginning. Most of the authors who write about it do not realize that in fact two theorems are dealt with and (for that reason or some other) when they decide to give historical comments, these comments are often inaccurate.

2) We believe that the evolution of the Orlicz-Pettis Theorem is worth detailed presentation. As well, we try to compile an exhaustive list of papers directly concerned with the subject matter of this article.

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3) Last but not least, the whole subject is not a closed chapter in mathematics. Some of the essays to follow will not only survey the existing material but will bring improvements into the area.

In short, our undertaking is on a different scale than the above mentioned talk by Kalton.

We dedicate this article to the memory of WLADYSLAW ORLICZ who died on August 9, 1990 in Poznań, Poland, at the age of 87. He was the last survivor of the Banach School of Functional Analysis that flourished in Lwów between the two world wars.

§1 THE ORIGINAL RESULT

There must exist (as always) a prehistory for the subject since the notion of the unconditional convergence of a series in a Banach space must have been motivated by some earlier considerations in concrete function spaces. There is a footnote to this effect in Orlicz's paper. We leave any investigation of such matters to true historians and take as our starting point the following definition and theorem (Orlicz [1929], §2, Satz 2; see also [1988]).

Definition 1.1. *Let X be a Banach space. A formal series $\sum_{n=1}^{\infty} x_n$ of elements in X is said to be* unconditionally convergent *("unbedingt konvergent") if for every permutation π of $\mathbb{N}$, the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ is convergent.*

Theorem 1.2. *Let X be a weakly sequentially complete Banach space. A series $\sum x_n$ is unconditionally convergent in X if (and only if) for each continuous linear functional $x^* \in X^*$*

$$\sum |x^*(x_n)|$$

is convergent.

REMARK. Equivalently, the theorem says that if $\sum x_n$ is weakly unconditionally Cauchy in X (i.e. given a weak neighborhood V of 0 in X, there exists $k \in \mathbb{N}$ such that for each finite subset e of $\{k+1, k+2, \dots\}$, we have $\sum_{n \in e} x_n \in V$), then it is (norm) unconditionally convergent. Of course, Orlicz knew that for scalar series unconditional convergence and absolute convergence are equivalent.

Here is the original proof of the theorem. The proof is translated from the German with modern ingredients introduced but without changing any essential point in the argument.

Proof. It will be sufficient to prove the following lemma.

Lemma. *If a sequence (u_n) in X is such that*

$$(1.1) \qquad \|u_n\| \le 1, \qquad n = 1, 2, \ldots \qquad \textit{and}$$

$$(1.2) \qquad \sum |x^*(u_n)| \quad \textit{is convergent for each } x^* \in X^*,$$

then

$$\lim_{n\to\infty} \|u_n\| = 0.$$

Suppose for the moment that the lemma has been shown. If our series were not unconditionally convergent, there would exist a sequence (v_i) ,

$$v_i = x_{\pi(n_i)} + x_{\pi(n_i+1)} + \cdots + x_{\pi(n_{i+1})},$$

such that

$$(1.3) \qquad \|v_i\| \ge \epsilon_0 > 0.$$

Setting $u_i = v_i/\|v_i\|$, the condition (1.2) would be still satisfied and $\|u_i\| = 1$ which contradicts the lemma.

The proof of the lemma. Let E be the closed linear span of (u_n). Since the weak topology in E is induced from the weak topology of X (Orlicz knew the Hahn-Banach Theorem; Banach published his version of it in the same first issue of Studia Mathematica) it can be assumed that $X = E$ and consequently that X is separable. We now show that:

For each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $x^* \in B^*$ (the unit ball of X^*)

$$(1.4) \qquad \sum_{n=N}^{\infty} |x^*(u_n)| \le \epsilon.$$

If not, there would exist $\epsilon_0 > 0$, $N_i \to \infty$, and a sequence (x_i^*) in B^* such that

$$(1.5) \qquad \sum_{n=N_1}^{\infty} |x_i^*(u_n)| \ge \epsilon_0.$$

Since X is separable, B^* is compact and metrizable in the weak* topology and so we can assume, by passing to a subsequence, that (x_i^*) is weak* convergent in B^*.

Let (a_i) be a bounded sequence (i.e. $(a_i) = a \in \ell^\infty$). For each $x^* \in X^*$

$$\sum_{n=1}^{\infty} a_n x^*(u_n) = \lim_{k\to\infty} x^*\left(\sum_{n=1}^{k} a_n u_n\right) = x^*(u) \tag{1.6}$$

(here the weak sequential completeness of X is used; the weak limit

$$u = \lim_{k\to\infty} \sum_{n=1}^{k} a_n u_n$$

could otherwise fail to be in X).

Since (x_i^*) is weak* convergent, one can treat the linear transformation

$$\begin{array}{rcl} \sum_{n=1}^{\infty} a_n x_1^*(u_n) & = & x_1^*(u) \\ \vdots & & \vdots \\ \sum_{n=1}^{\infty} a_n x_p^*(u_n) & = & x_p^*(u) \\ \vdots & & \vdots \end{array}$$

with the matrix $\{x_p^*(u_q)\}$ as a coercive summability method ("lineare konvergenzerzeugende Limitierungsmethode") which to every bounded sequence (a_n) associates the limit (i.e. $\lim_{i\to\infty} x_i^*(u)$). By a theorem of Schur ([1921] Satz 3) the matrix $\{x_p^*(u_q)\}$ must have the following property: For each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $p = 1, 2, \ldots$

$$\sum_{n=N}^{\infty} |x_p^*(u_n)| \le \epsilon. \tag{1.7}$$

This contradicts (1.5). Then (1.4) holds, which implies the lemma.

§2 The Orlicz–Pettis Theorem

According to Orlicz [1955] and [1971], it was an analysis of the above proof of Theorem 1.2 that motivated the introduction of the notion of subseries convergence. By assuming in Theorem 1.2 weak subseries convergence instead of the weak unconditional Cauchy condition, essentially the same proof gives another theorem. The new theorem was presented during a meeting of the Scientific Associations of Students of Mathematics and Physics, Lwów 1931 (Orlicz [1955], footnotes 2 and 5[1]; [1988]), and is recorded in Banach's "Théorie des opérations linéaires" in the last section of "Remarques". We stress that the theorem is there openly credited to Orlicz, although no proofs were given for the results discussed in the "Remarques".

Theorem 2.1 ([1932], p. 240). *In a Banach space X the following are equivalent:*

1. *The series $\sum x_n$ is unconditionally convergent.*
2. *The series $\sum x_n$ subseries convergent.*
3. *The series $\sum x_n$ is weakly subseries convergent.*

Here subseries convergence of $\sum x_n$ means that for each subsequence (x_{n_i}) of (x_n) the series $\sum x_{n_i}$ is convergent. As Orlicz says [1971], the theorem was never published separately with a proof since the old proof worked fine.

Let us examine the last statement more closely: When one analizes the original proof of Theorem 1.2, one can notice immediately that the proof can actually be simplified. Instead of formulating the lemma, it is sufficient to proceed by contradiction only once and show that ((1.3) above)

$$\|v_i\| \geq \epsilon_0 > 0$$

cannot happen. For this purpose the proof of the lemma as given above can be repeated (with $u_n = v_n$) *assuming presently that $\sum v_n$ is weakly subseries convergent.* The Theorem of Schur must now be used with its

[1] [1955] (translated from the Polish):

footnote 2. Unconditionally convergent series in Banach spaces appear first in Orlicz [1929]. The equivalent notion of subseries convergent series was introduced by the author in a Communication done during a meeting of the Scientific Associations of Students of Mathematics and Physics in Lwów in 1931; compare also Banach [1932], p. 240.

footnote 5. Theorem 3B was first published in author's paper [1929]; part A of the theorem was contained in author's Communication during a meeting ... (compare footnote 2). S. Banach gave the main theorems of this Communication without proof in his monograph [1932], p. 240.

full force (see Schur [1921], Remark after the proof on p. 90), that is, taking into account that (1.5) already works when $a = (a_n)$ is such that $a_n = +1, -1$, or 0 (and not all $a \in \ell^\infty$ as previously). Then the analogue of (1.5) still holds in view of the weak subseries convergence of $\sum x_n$ (and so X need no longer be assumed weakly sequentially complete). The Theorem of Schur (in its strong form) shows that (1.3) holds, i.e. that

$$\sum_{n=N}^{\infty} \|u_n\| \leq \epsilon$$

and this ends the proof of Theorem 2.1.

It is thus beyond any doubt for us (and we believe for any mathematician reading these words) that indeed "the old proof worked fine" once the assumption of weak subseries convergence replaced the weak unconditional Cauchy condition in the original Theorem 1.2.

Let us add immediately that there is a similar remark in Bessaga and Pełczyński ([1958], footnote 10).

§3 THE PETTIS' CONTRIBUTION

Let us start by comparing the two theorems. As already noticed by Orlicz in [1929] (example after Satz 3) his Theorem 1.2 does not hold in every Banach space. The most familiar example nowadays is provided by $\sum e_n$ in c_0 (where $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with the "1" in the n-th position). It is clear that $\sum e_n$ is weakly unconditionally Cauchy but not norm convergent therein.

On the other hand if X is weakly sequentially complete, weakly unconditionally Cauchy series concide with weakly subseries convergent series. Thus not only does Theorem 1.2 imply Theorem 2.1 but also vice versa.

The difference between the two theorems becomes striking when they are properly formulated in the language of vector measures.

First we observe that Theorem 1.2 is equivalent to (Orlicz [1929], Satz 3)

Theorem 1.3. *Let X be a weakly sequentially complete Banach space and suppose that $\sum x_n$ is perfectly bounded in X (i.e. the set $\{\sum_{n \in e} x_n : e \in \mathcal{F}(\mathbb{N})\}$ of all unordered finite partial sums is a bounded set in X). Then $\sum x_n$ is convergent in X.*

Its vector measure formulation is as follows:

Theorem 1.3′. *Let X be a weakly sequentially complete Banach space, $\mathcal{R}$ a ring of sets and $\mu : \mathcal{R} \to X$ an additive set function. Then μ is bounded if and only if it is exhaustive.*

We recall that μ is said to be *bounded* if its range $\mu(\mathcal{R})$ is a bounded set in X, and is said to be *exhaustive* if for each disjoint sequence (E_i) of sets in $\mathcal{R}$,

$$\lim_{i\to\infty} \|\mu(E_i)\| = 0.$$

An equivalent formulation of 1.3′, even closer to 1.2, would be that weak exhaustivity of μ implies its exhaustivity.

On the other hand, the vector measure form of Theorem 2.1 is:

Theorem 2.1′. *Let X be a Banach space, $\mathcal{R}$ a σ-ring of sets and $\mu : \mathcal{R} \to X$ a weakly countably additive set function. Then μ is (norm) countably additive.*

Pettis, writing [1938], knew Orlicz's [1929] paper, and he knew Theorem 2.1 from Banach's book. However he did not realize that the original proof of Theorem 1.2 works for Theorem 2.1. He writes (Pettis [1938], p.231):

"The next lemma and theorem were proved by Orlicz ([1929], Theorem 2) for the case X weakly sequentially complete; the general theorem is credited by Banach ([1932], p.240) to the same author without proof or reference. Since we know no published proof to which to refer, and since the lemma is fundamental for our purposes, we include the following demonstration."

Orlicz Lemma (Pettis [1938], Lemma 2.31). *If $\sum x_n$ is weakly subseries convergent in a Banach space X, then given $\epsilon > 0$ there exists N_ϵ such that $x^* \in X^*$ and $\|x^*\| = 1$ implies*

$$\sum_{n=N_\epsilon}^{\infty} |x^*(x_n)| < \epsilon;$$

hence $\|x_n\| \to 0$.

Theorem (Pettis [1938], Theorem 2.32). *In a Banach space X weak and norm subseries convergence are equivalent.*

The derivation of the Theorem from Orlicz Lemma is standard; it is clear that the lemma corresponds to (1.7) in our version of Orlicz's proof of Theorem 1.2. What was its proof by Pettis? Here is the surprise (or is it?):

> ***The proof is nothing more than a polished Banach space version of Orlicz's original proof.***

Let us explain. In the language of Banach space theory, an interpretation of the Theorem of Schur that was invoked by Orlicz is that weak and norm convergences of sequences are the same in ℓ^1. (Today this is called the Schur property of ℓ^1.) Banach ([1932], pp. 138-139) gives a direct proof of the latter result (referring to Schur on p.239 of [1932]) and, as one can guess, the proof in Banach's book also makes it clear that, to get the result, instead of the whole dual ℓ^∞ of ℓ^1, one can use sequences taking only the values $+1, -1, 0$. (Compare with Schur's remark in [1921] referred to in §2.) Pettis, using the proof by Banach (he refers to the proof and not the statement of the "Schur property"), gives an elegant derivation of the "Orlicz Lemma" that is entirely in the spirit of the modified proof of Orlicz.

§4 FINALE

For further reference, we will give the schematic name The First Orlicz Theorem to Theorem 1.2 *and* its equivalent form in Theorem 1.3

When looking back at The First Orlicz Theorem and knowing the subsequent research that has been motivated by this theorem, the statement under 1.3 appears to be more important than 1.2; the Revisited Theorem (1.3) goes more visibly into the "heart of the matter" in the sense that its formulation requires only the linear topological notion of boundedness and thus is suitable for further generalizations. On the other hand, the statement of The First Orlicz Theorem (1 .2), though rather accidental in the sense that its form is made possible by the "accidental" existence of the separating dual (= weak topology) on the space, has the merit of provoking the "right" statement in the form of The Second Orlicz Theorem (Theorem 2.1) or, as it is called today, The Orlicz–Pettis Theorem.

CONCLUSION. We can now pinpoint the source of misunderstandings concerning the history of The Orlicz–Pettis Theorem as follows.

Orlicz proved in [1929] a general theorem (The First Orlicz Theorem) which happens to be equivalent in weakly sequentially complete Banach spaces to another statement that, in turn, in this new form generalizes to arbitrary Banach spaces (The Second Orlicz Theorem or The Orlicz–Pettis Theorem).

Perhaps Pettis was being overly cautious in his remarks concerning The First Orlicz Theorem, or perhaps he was unaware of the essential generality of Orlicz's proof. In any case there is a natural tendency to interpret Pettis' comment quoted above as saying that Orlicz proved in [1929] *exactly* (as opposed to ***in particular***) "The Orlicz–Pettis Theorem in weakly sequentially complete Banach spaces"; cf., e.g., McArthur [1967] (and many other authors following him) who had fallen into this trap. Had Pettis added the innocent "in particular" in a proper place, the misunderstandings in question, most probably, would have been avoided.

Let us add immediately that although The Orlicz–Pettis Theorem was formulated and proved in full generality by Orlicz himself, we do feel that the name "Orlicz–Pettis Theorem" is not only convenient, but there is a genuine reason to keep it the way it is: Today The Orlicz–Pettis Theorem appears most often as a theorem about vector measures (2.1′), and Pettis was the first to state it in this form.

We finish by mentioning a natural problem connected with this early stage of our discussion of The First Orlicz Theorem and The Orlicz–Pettis Theorem. The answer to this problem, although contained in Bessaga–Pełczyński ([1958], C.14), seems to remain largely ignored by the general public. In order to clear the air before stating the problem let us recall a few facts about the interdependence between subseries convergence and unconditional convergence.

In a Hausdorff topological Abelian group X, the subseries convergence of $\sum x_n$ implies its unconditional convergence. To see this, it suffices to consider $\sum x_n$ in the completion $\overline{X}$ of X and apply the Cauchy condition for summability.

If X is sequentially complete, the unconditional convergence of $\sum x_n$ implies its subseries convergence, but there is no reason to expect this implication to remain valid without any completeness condition on X. An easy confirmation of that feeling (communicated to us by Z. Lipecki) can be obtained this way:

Let $(X, \|\cdot\|)$ be an F-space and let (x_n) be an arbitrary sequence of independent vectors in X such that $x_n \to 0$. Choose a subsequence (y_k) of (x_n) such that $\sum \|y_k\| < \infty$. Then $\sum y_k$ is subseries convergent. Let Y be the $\aleph_0$-dimensional subspace of X generated by the vectors $\{\sum_{k=1}^{\infty} y_k, y_1, y_2, \dots\}$. Then $\sum y_k$ is unconditionally convergent in Y. However, it cannot be subseries convergent in Y in view of Corollary 1 in Labuda–Lipecki [1982].

Now let X be again a Banach space. It could still be possible, perhaps, that the weak unconditional convergence of $\sum x_n$ in X would imply its norm unconditional (whence subseries) convergence. Of course, this implication *is* valid in an X in which The First Orlicz Theorem holds. Such Banach spaces are characterized by Pełczyński [1957] (see Bessaga–Pełczyński [1958], Theorem 5, for the first published proof) as those which do not contain c_0.

Here is an example which disproves the above guess in c_0: Consider again the standard basis (e_n), and set $x_n := e_n - e_{n+1}$. Then $\sum x_n$ is not norm convergent in c_0, but for every permutation π of $\mathbb{N}$, $\sum x_{\pi(n)} = e_1$ holds with respect to the weak topology.

Thus the class of Banach spaces in which weak unconditional convergence implies unconditional convergence is contained in the class of Banach spaces which do not contain c_0, and is therefore equal to the class of Banach spaces in which the weak unconditional Cauchy condition is sufficient to guarantee unconditional convergence. This result remains true in sequentially complete locally convex spaces.

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