Real Analysis Exchange Vol 16 (1990-91)

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Three Methods of Constructing ω-Limit Sets[1]

1. Introduction

Our main objective is to discuss three methods of constructing ω-limit sets for functions $f: [0,1] \mapsto [0,1]$. A set $K \subset [0,1]$ is called an ω-limit set for f if there exists $x_0 \in [0,1]$ such that K is the cluster set of the sequence $(x_n) = (f^n(x_0))$. As usual $f^1 = f$ and $f^{n+1} = f \circ f^n$, $n = 1, 2, 3, \ldots$. We write $\omega_f(x_0) = K$ to indicate K is the ω-limit set of x under f.

Let's begin by presenting a bit of background material that will be relevent to our discussion. First, for $\mathcal{C}$, the class of continuous functions, every ω-limit set is either a compact nowhere-dense set, or consists of a finite union of closed intervals. Furthermore, each nonempty set satisfying one of these two conditions can be realized as an ω-limit set for some $f \in \mathcal{C}$. [ABCP], [BS].

For sufficiently well-behaved continuous functions one finds that either there is a single set that serves as ω-limit set for almost all $x \in [0,1]$, or there is some form of chaos (or both). For the typical continuous f, however, there will exist continuum many pairwise disjoint nowhere-dense perfect sets that collectively serve as ω-limit sets for almost all $x \in [0,1]$. A bit more precisely, there exists a set of full measure M, such that for every $x \in M$, $\omega_f(x)$ is a Cantor set K_x, and if $x, y \in M$, $x \neq y$, then $K_x \cap K_y = \emptyset$ [ABL].

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[1] This paper is an expanded version of a talk presented at the FOURTEENTH ANNUAL SUMMER SYMPOSIUM IN REAL ANALYSIS, SAN BERNARDINO

Now let $\Lambda(f)$ denote the collection of ω-limit sets of f. Given a nonempty family $\mathcal{K}$ of nonempty compact sets it is natural to ask whether there exists $f \in \mathcal{C}$ such that $\mathcal{K} = \Lambda(f)$, or $\mathcal{K} \subset \Lambda(f)$. Various well-known results indicate restrictions on a family $\mathcal{K}$ if it is to satisfy $\mathcal{K} = \Lambda(f)$ for some $f \in \mathcal{C}$. For example, if $\mathcal{K}$ contains a 3-point ω-limit set, it must contain n-point ω-limit sets for every positive integer n. Similarly, a finite ω-limit set cannot be a proper subset of any other finite ω-limit set for $f \in \mathcal{C}$. (If one drops the continuity requirement entirely, however, there are no restrictions. Given any nonempty family $\mathcal{K}$, there exists a measurable function f for which $\mathcal{K} = \Lambda(f)$ [BCP]).

There are also some positive results. For example, given any nonempty family $\mathcal{K}$ of nowhere-dense compact sets in [0,1], there exists a continuous $f: [0,1] \to [0,1]$ such that to each $K \in \mathcal{K}$ corresponds a set $K^* \subset \left[\frac{1}{3}, \frac{2}{3}\right]$ that is homeomorphic to K, and a number $x \in [0,1]$ such that $\omega_f(x) \cap \left[\frac{1}{3}, \frac{2}{3}\right] = K^*$ [ABCP].

We shall discuss some methods that seem to be usefull in constructing ω-limit sets or families of ω-limit sets for functions, but our emphasis will be on functions in the larger class $\mathcal{DB}_1$ of Darboux functions in the first class of Baire. This class arises naturally in connection with Newton's Method. We discuss this briefly.

2. Newton's Method

Newton's Method for estimating the zeros of a differentiable function f entails iterations of the function $g(x) = x - \frac{f(x)}{f'(x)}$. It is more convenient to consider a function defined on all of $\mathfrak{R}$ here since iterations may well take one out of any predetermined interval I. And some conventions must be adopted when an iterate hits a point at which $f'(x)$ vanishes. Even for polynomials of degree > 3, some sort of "random" behavior is possible for orbits of certain points and, when all the roots are real, a Cantor set of points will "escape" $\mathbb{R}$, that is, the orbits of points in that set hit the set on which f' vanishes.

Now, Newton's method makes sense for any differentiable function, not just for $\mathcal{C}^1$ functions. The function g need not be continuous. It must be in $\mathfrak{DB}_1$, however, on any interval on which f' does not vanish. This suggests studying the iterative behavior of functions in $\mathfrak{DB}_1$ or in certain subclasses of $\mathfrak{DB}_1$ that are larger than $\mathcal{C}$.

3. Constructions:

Let $\mathcal{K}$ be a nonempty family of compact subsets of $I = [0,1]$, and let $\mathfrak{F}$ be a family of functions from I to I. For $f \in \mathfrak{F}$, let $\Lambda(f)$ denote the class of ω-limit sets for f. For $\mathcal{K}$ and $\mathfrak{F}$ given, does there exist $f \in \mathfrak{F}$ such that $\mathcal{K} \subset \Lambda(f)$? We discuss three methods that have proved useful in approaching this problem and its varients: A - Arithmetic methods; B - Interval-orbits, and C - Specifying orbits.

A - Arithmetic Methods.

We illustrate this method by constructing functions whose iterative patterns are built into the ternary representations for numbers in I. Let C be the ordinary Cantor set, let $K = \frac{1}{3}C + \frac{1}{3}$, and let S consist of those $x \in I$ with expansions of the form $x = .10A_1 1010A_2 1010A_3 10...$ where each A_i is a block of 0's and 2's. Let $g(x) = 1 - 3\,\mathrm{dist}(x, K)$. The graph of g over the complement of K consists of line-segments of slope $\pm$ 3. On $\left[\frac{1}{3}, \frac{2}{3}\right]$, these segments form spikes on the intervals contiguous to K. On K itself, $g = 1$.

Intuitively, one sees that while all points whose orbits land on K will be absorbed by 0 two iterations later, there will be many points that repeatedly narrowly miss this fate: they will land near K, escape near 1, then near 0, then continually triple in value until they are again near K, and on and on. One sees this behavior for all $x \in S$. Indeed, simple arithemetic shows that for $x = .10A_1 1010A_2 1010A_3 10...$, the smallest integer n_1 such that $g^{n_1}(x) \in \left[\frac{1}{3}, \frac{2}{3}\right]$ leaves $g^{n_1}(x) = .10A_2 1010A_3 10....$

In effect, the first block A_1 has been dropped. If M_1 is any nonempty closed subset of $K \cap \left[\frac{1}{3}, \frac{4}{9}\right]$ we can choose the blocks A_k so that the values $.10A_k 1010A_{k+1} 10...$ approximate all points in M_1 and no others, thus the part of the orbit of x that lies in $\left[\frac{1}{3}, \frac{4}{9}\right]$ clusters exactly on M_1, i.e. $\omega_g(x) \cap \left[\frac{1}{3}, \frac{4}{9}\right] = M_1$.

We cannot use this idea to obtain a continuous f that realizes a homeomorphic copy of every nonempty nowhere-dense set as an ω-limit set. But we can extend the "first return map" of g that maps S onto S to a function G in $\mathcal{DB}_1$ that maps $\left[\frac{1}{3}, \frac{4}{9}\right]$ onto itself. In fact using a theorem in [PL], we can choose G to be approximately continuous, (and therefore a derivative). Thus F does realize every nowhere-dense compact set (up to homeomorphism) as an ω-limit set.

B - Interval-Orbits.

Suppose we believe a function f has a certain set M as an ω-limit set. We wish to prove the existence of an $x \in I$ such that the sequence $\{ f^n(x) \}$ has M as its cluster set. We might proceed as follows. We find an interval K_0 whose orbit approximates M at least for a while. Inside K_0 we find a closed interval K_1 whose orbit approximates M for a longer while and for which the error tolerance is smaller. We continue in this manner obtaining a nested sequence of intervals whose intersection does the job.

A recent example of this method [BS] may serve as an illustration. Suppose M is an uncountable nowhere-dense compact set. (The reader may wish to use the set $\{0, 1\} \cup \bigcup_1^\infty \frac{M_1}{3^{n-1}}$, of the previous section, as a model with $M_0 = \{1\}$, $\{a_0\} = \{0\}$, and $M_n = \frac{M_1}{3^{n-1}}$.) Let a_0 be a condensation point of M. One shows first that one can express M as a disjoint union of compact sets $M = \{a_0\} \cup \bigcup_{n=0}^\infty M_n$, in such a way that there is a function $f: I \to I$ with the following properties:

(i) $f(M_n) = M_{n-1}$ for all $n>0$

(ii) $f(M_0) = \{a_0\}$

(iii) For every $u \in M$ and any neighborhood U of u, $f(U)$ is a neighborhood of $f(u)$.

One then shows that given u, v $\in$ M, U a neighborhood of u, and $\partial>0$, there is a closed interval $K_0 \subset U$ and a positive integer n such that $f^n(K_0)$ is a neighborhood of v and $f^i(K_0)$ is in the ∂-neighborhood of M for all $i \le n$. Letting $f^n(K_0)$ play the role of U, decreasing ∂, and choosing an appropriate subinterval K_1 of K_0, one finds the orbit of K_1 reaches another point w $\in$ M in a finite number of steps, passing near v on the way. One continues inductively in the obvious manner and shows that the intersection of the nested sequence of closed intervals K_n obtained is a point having M as ω-limit set.

Varients involving countable compact sets M lead to the general result that every nonempty nowhere-dense compact set $M \subset I$ is an ω-limit set for some $f \in \mathcal{C}$ [BS].

The technique has also been useful for proving that a function in $\mathcal{DB}_1^*$ can achieve certain combinations of sets as ω-limit sets. In this connection, Keller [K] has shown that any <u>individual</u> set M that is an ω-limit set for an $f \in \mathcal{DB}_1^*$ is also an ω-limit set for some continuous function. But the discontinuities allowable for functions in $\mathcal{DB}_1^*$ permit one to obtain combinations of ω-limit sets not possible for continuous functions. Intuitively, a discontinuity point can serve as a "distribution" point for intervals near it. For example, $f(x) = \left|\sin \frac{\pi}{x}\right|$, $(f(0)=0)$ has for each subset A of $\left\{\frac{1}{n} : n=1,2,3,\ldots\right\}$ the set $\{0\} \cup A$ as an ω-limit set. One simply chooses intervals in the process described above whose orbits alternate between being near 0 and being near points of A.

Keller has used a refined version of this argument to show that there is a function f in $\mathcal{DB}_1^*$ (in fact, with only one discontinuity point) that has a homeomorphic copy of every nonempty, nowhere-dense closed set as an ω-limit set. In addition, for each n = 1, 2, ... , there is a set consisting of n pairwise disjoint intervals that is in $\Lambda(f)$. This improves a result in [BCP] where a function in $\mathcal{DB}_1$ was found with the same properties.

Keller's example has homeomorphic copies of all nonempty closed subsets of a Cantor set P as ω-limit sets. A similar construction gives rise to an $f \in \mathcal{DB}_1^*$ that has <u>each</u> such subset as an ω-limit set. But such an f must be discontinuous at each point of P. To achieve even all doubleton sets of the form {0,p} and {1,p} (p $\in$ P) as ω-limit sets, the oscillation of f must be 1 at each p of P.

How well does this technique carry over when we deal with functions in $\mathcal{DB}_1$ that that are not necessarily in $\mathcal{DB}_1^*$? We're not sure. Suppose we wanted to construct an $f \in \mathcal{DB}_1$ that has a certain collection of sets as as ω-limit sets. Perhaps we conceive a candidate by visualizing where the distribution points should be. Each distribution point has to be a point of discontinuity of the function.

If the set is dense in some interval I, our previous argument has problems. We can't be sure we can map a small interval in I onto a small interval as needed. Perhaps we can choose an appropriate set H, residual in some small interval, and require that f(H) be residual in a small interval where needed. Perhaps an argument using relative intervals of a residual set can be carried out. But what can we say about the intersection of the nested sequence of relative intervals that arises? How can one be sure it is not empty? And, if it is nonempty, how can we be sure the point it defines has the desired orbital behavior? Perhaps some form of the Banach Mazur game could be usefully applied to the convex hulls of the relative intervals of H. We tried, without success, to use such an idea to find an $f \in \mathcal{DB}_1$ which had hoeomorphic copies of every compact (nonempty) subset of I as ω-limit sets.

The difference between our earlier examples and the example sought here is that we wish to consider also those sets that have interiors but are not just finite unions of intervals. No function in $\mathcal{DB}_1^*$ can have such a set as an ω-limit set, but functions in $\mathcal{DB}_1$ can. It is still an open problem to determine whether a "universal" function $F \in \mathcal{DB}_1$ exists. (Universal up to homeomorphism, of course).

C - Specifying orbits.

Let's view the idea of the method of the previous section in the following way. We choose an interval I whose orbit for a while moves near the desired ω-limit set. Then we improve matters by getting a subinterval of I whose orbit moves nearer the desired ω-limit set for a longer while. In the limit, we obtain our desired point whose orbit does what we want.

If we are trying to find a function with the desired ω-limit set, why not try to get right to the heart of things - pick the orbit, or orbits, we want, and see if we can find a function in the desired class ($\mathcal{C}$, $\mathcal{DB}_1^*$, $\mathcal{DB}_1$, etc.) that has points with these orbits. This approach has been used with some success in

[ABCP]. One finds a condition on a sequence $S = \{x_n\}_0^\infty$ that guarantees the existence of a function $f \in \mathcal{C}$ such that $f(x_n) = x_{n+1}$ for all $n = 0, 1, 2, \ldots$. This condition was then used to prove parts of the theorem that asserts that each nonempty, nowhere-dense closed set M is an ω-limit set for some $f \in \mathcal{C}$.

For the family $\mathcal{DB}_1$ we already know that every nonempty closed set $K \subset I$ is in $\Lambda(f)$ for some $f \in \mathcal{DB}_1$. We wish to address the question: "What families $\mathcal{K}$ are contained in $\Lambda(f)$ for some $f \in \mathcal{DB}_1$?" The specifying of orbits has proved useful in attacking the problem when $\mathcal{K}$ is countable. The basic tool is the theorm below.

Theorem: *Let D be a denumerable subset of I, and let $f: D \to D$. For $\varepsilon > 0$, let $D_\varepsilon = \{x \in D: \mathrm{osc}\,(f,x) \geq \varepsilon\}$. If the closure of D_ε is denumerable for every $\varepsilon > 0$, then f can be extended to a function $F \in \mathcal{DB}_1$ on I.*

The theorem is used in the following way. One wishes to find $f \in \mathcal{DB}_1$ that has each of the sets $K_1, K_2, \ldots$ as ω-limit sets. One tries to find sequences $S_1 = \{x_k^1\}, S_2 = \{x_k^2\}, \ldots$ such that for each $n = 1, 2, 3, \ldots$ S_n has K_n as cluster set, and such that the function f defined on $\bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \{x_k^n\}$ by $f(x_k^n) = x_{k+1}^n$ satisfies the condition of the theorem.

For example, if $\cap K_n \neq \emptyset$, say $b \in \cap K_n$, one can construct sequences S_n that meet the condition rather easily. The point b serves as a "distribution" point for all the sets K_n. Replacing one "distribution" point by two, however, causes the result to fail. For example, the collection $\mathcal{K}_1$ of two-point sets $\{0, r\}$, r rational, is in $\Lambda(f_1)$ for some $f_1 \in \mathcal{DB}_1$. Similarly, the collection $\mathcal{K}_2$:{r, 1} is contained in $\Lambda(f_2)$ for some $f_2 \in \mathcal{DB}_1$. But the collection $\mathcal{K}_1 \cup \mathcal{K}_2$ isn't: if $\mathcal{K}_1 \cup \mathcal{K}_2 \subset \Lambda(f)$ for some f, then f must be discontinuous everywhere.

Keller [K] has obtained a number of conditions that imply that a countable family $\mathcal{K}$ is contained in $\Lambda(f)$ for some $f \in \mathcal{DB}_1$. But a characterization of such families has not yet been obtained.

We end by mentioning that the theorem in this section can be extended to sets D that are nondenumerable. In general, one can then conclude only that the extended function F is a Baire 1 function. If, however, the closure of D has measure zero, F can be chosen to be approximately continuous. Thus, the methods of this section sometimes can be applied to deal with uncountably many specified orbits. As an illustration, consider the set S in section 3A. Each $x \in S$ has the representation $x = .10A_1 1010A_2 1010A_3 10\ldots$. We can define orbits for all $x \in S$ by defining $f(x) = .10A_2 1010A_3 10\ldots$. The resulting function maps S onto S and is continuous on S. Since $S \cup \left(K \cap \left[\frac{1}{3}, \frac{4}{9}\right]\right)$, the closure of S, has measure zero, f can be extended to an appoximately continuous F on I. The function F has each nonempty closed subset of $K \cap \left[\frac{1}{3}, \frac{4}{9}\right]$ as an ω-limit set. Observe that f is simply the first-return map of the function g that we discussed in section 3A.

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Received October 21, 1990