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Some interpolation problems in real and harmonic analysis[1]

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Suppose we have any function space F and a subspace G of "good" functions. For arbitrary $f \in F$, we wish to find $g \in G$ which coincides with f on some set E. This is an interpolation problem. It is necessary to distinguish between two variants of the problem.

In the first case, *interpolation with fixed knots*, a set E (not necessarily finite) is given a priori. If the problem is resolvable for every $f \in F$ we say E is an *interpolating set* for the pair (F, G).

In the second case, E is not given and it can be chosen, depending on f, in such a way that it may be "thick" in a metric sense or in cardinality. This is *free interpolation*. An elementary example of the first problem is polynomial interpolation by the Lagrange or Newton method. Another and deeper example is given in the famous Rudin-Carleson theorem on the disc-algebra of functions.

An important example of the second type of problem is the famous Menshoff "correction" theorem in Fourier analysis. In what follows I will be concerned with some aspects of interpolation of continuous functions arising in classical and harmonic analysis and I will describe recent progress and some open questions.

I. Interpolation by smooth functions.

Here we deal with the following problem: to what degree can one improve the smoothness of given function $f: I = [0, 1] \to \Re$ by free interpolation on perfect (nonempty) sets.

The history of this question begins with a curiosity. In the mid-thirties Ulam conjectured that for every $f \in C(I)$ one can define an analytic function g which coincides with f on a perfect set.

One might observe in favor of this conjecture that if f is "bad", say nowhere differentiable, or it has no interval of monotonicity, then some level sets of f are uncountable, so we can put $g \equiv const$.

[1] This is a summary of a talk given at the Fourteen Summer Symposium on Real Analysis at San Bernardino.

In his well-known monograph [19] Ulam stated that the conjecture was proved by Zahorskii [18] and then he discussed the possibility of high dimensional generalizations. But in reality, Zahorskii had given a negative answer: he had constructed a function $f \in C^{\infty}(I)$ which has at every point $x \in I$ the radius of convergence of the Taylor series equal to zero. It is clear that this function gives a counterexample to Ulam's conjecture. It should be mentioned that the first example of such a function was constructed by H. Cartan [5].

Nevertheless many years later the problem led to the new and interesting developments.

The comprehensive article on this theme was published by Agronski, Bruckner, Laczkovich and Preiss [1], see also [2], [3], [12]. The following proposition is true.

<u>Theorem 1</u>: *Let* $f \in C(I)$. *Then*:

(i) [1] *There exists a function* $\varphi \in C^{\infty}(I)$ *which coincides with* f *on an infinite set of points.*

(ii) [12] *There exists a perfect set* $E \subset I$ *such that the restriction* $f|_E$ *is* C^{∞} *relative to* E.

(iii) [1] *There exists a function* $g \in C^1(I)$ *such that the set* $\{x: f(x) = g(x)\}$ *is uncountable.*

The last assertion follows from (ii) and Whitney's extension theorem.

The authors mentioned ([1], p. 660, [2]) that the problem of extension of (iii) to $g \in C^2(I)$, or $g \in C^{\infty}(I)$, remains unsolved. Recently Buczolich obtained a theorem which seemingly came up close to the case $g \in C^2$:

<u>Theorem 2</u> [4]. *For every* $f \in C(I)$ *there exists a convex* g, *which interpolates* f *on some perfect set.*

Nevertheless the answer to the problem above is "NO". Free interpolation of a continuous function by a twice smooth function on a perfect set is, in general, impossible. We state this theorem in slightly stronger form. Let us say $f \in C^{\alpha}(I)$, $[\alpha] = k$, $\alpha \neq k$, if $f \in C^k(I)$ and $f^{(k)}$ belongs to the Hölder space $H^{\alpha-k}(I)$. The following proposition is true.

Theorem 3. *There exists a Lipshitz function* f: I $\to \Re$ *such that for every* $g \in C^{\alpha}(I)$, $\alpha > 1$, *the set* $\{x: f(x) = g(x)\}$ *is countable (with only a finite set of accumulation points).*

We can give an explicit formula for *f*. Let *r(x)* equal *1* for $0 < x < 1/2$, equal *-1* for $1/2 < x < 1$ and be periodically extended on $\Re$. Then for small enough $q \in (0, 1)$ and rapidly increasing integers $\{\nu_n\}$ the function

$$f(x) = \int_0^x \sum_{n \in N} q^n r(\nu_n t) dt \tag{1}$$

satisfies the statement of Theorem 3.

Remarks.

R.1. One can show that the f just defined possesses the following property: for every perfect $E \subset I$ the second divided difference $\omega_2 f$ is unbounded on any (nonempty) portion of E. So $\omega_3 f$ does not preserve the sign on E. This gives an answer to another problem from [1] (p. 677).

R.2. It is well known that Rademacher series and lacunary trigonometric series often have the same behavior in many problems and they are used equally in the construction of various counterexamples. But if we replace the Rademacher functions in (1) by harmonics $\cos \nu_n t$ then f will allow the interpolation by a twice smooth function on perfect set. This follows from the theorem below.

We see from Theorem 3 that when an order of smoothness of given function f increases from zero to one the possibility of "improvement of smoothness" by free interpolation on perfect sets vanishes. Even if f is "almost C^1" we can't jump over the value *1* in the scale of smoothness. It is clear (by repeated integration) that the same effect appears near every integral order.

Let now $f \in C^1(I)$. Can one improve the smoothness? The answer is "yes".

Theorem 4. *For every* $f \in C^1(I)$ *there exists a function* $g \in C^2(I)$ *such that* $f \equiv g$ *on some perfect set* $E \subset I$.

From the previous it follows that this is the best possible.

Now let us move further on scale of smoothness on $f \in C^2(I)$. We might expect to improve its smoothness one order. But at this point the last new effect is waiting for us.

Theorem 5. *There exists* $f \in C^2(I)$ *such that for each* $g \in C^3(I)$, *or even* $g \in C^{2+\varepsilon}(I)$, *the set* $\{f \equiv g\}$ *is countable.*

Of course the same holds for $f \in C^k$, $k > 2$: in general this function allows the free interpolation by $g \in C^{k+\varepsilon}$ only on countable sets.

It's worth mentioning that in theorems 3 and 5 we run into "exceptional" effects. A typical (in the Baire category sense) function has a more respectable behaviour:

Theorem 6. *For every* k, *a typical function* f *from space* $C^k(I)$ *can be interpolated on suitable perfect set by* $g \in C^\infty(I)$.

Two remarks at the end of this part.

R.3. In [1] the authors showed that if a graph f has only finite intersection with graph each polynomial, then it has arbitrary high smoothness on some subintervals. They asked if it is true that if it has finite intersection with every analytic function on I, then f is C^∞ on some subinterval. A counter-example was constructed by Fjodoroff [6] who used some perturbation of the Cartan-Zahorskii example. Let f_1 be a C^∞ function with radius of convergence of its Taylor series equal to zero at every point. Further

$$f_2(x) = \sum_{n\in N} \frac{1}{n!} \sum_{k=1}^{2^{n-1}} \theta\left(x - \frac{2_{k-1}}{2^n}\right)\left(x - \frac{2_{k-1}}{2^n}\right)^n,$$

where

$$\theta(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}.$$

Then $g = f_1 + f_2$ has a finite intersection with every analytic g and it is not C^∞ on any interval.

R.4. In all the above presentation we described "thickness" of the interpolation set only in power terms. It is natural because metrical conditions even in the most weak sense could not be required: Typical $f \in C(I)$ may coincide with $g \in C^1(I)$ only on a set of Hausdorff measure zero (relative to any given generating function *h*) [6, 7].

As distinct from this, in what follows the metrical approach plays the main role.

II. Interpolation in Fourier analysis.

We are going to interpolate any continuous function *f* on the circle $T = \Re/2\pi$ by functions with "well behaved" Fourier expansion

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int} \tag{2}$$

This good behaviour we indicate by belonging to important spaces *U(T)* and *A(T)*. For more details on this topic see {16, 17}.

II.1. *U(T)* is the Banach space of functions *g* which are expandable in uniforrmly convergent series (2) with norm

$$\| f \|_v = \sup_N \| S_N(g) \|_{C(T)}, \; S_N = \sum_{|n| \le N} \hat{g}(n) e^{int} \tag{3}$$

The problem of free interpolation leads to the following.

Menshoff "correction" theorem". *For every* $f \in C(T)$ *and every* $\varepsilon > 0$ *there exists* $g \in U(T)$ *such that* mes $\{t: f(t) \neq g(t)\} < \varepsilon$.

The proof is constructive; you can see the simple version of the proof, for example, in [16]. Unexpectedly, investigation of interpolating compacta for pairs (C, U) turned out to be an essentially a more difficult problem. It is well known that such a compactum is necessarily of measure zero. The converse assertion was proposed in [14] (p. 89) and it was proved by Oberlin [13].

Theorem [13]. *Let* E ∈ **T** *is a compactum of measure zero and* f ∈ C(E). *Than it can be extended on the circle as element of* U(**T**).

It may seem shocking at first glance that the proof is based on very powerful tool - the famous Carleson theorem on convergence a.e. of L^2 - Fourier series. It is unknown if one can avoid this and give a constructive proof. Detailed investigation of relations between these two theorems seems to be an interesting problem. It should be mentioned that the two dimensional analogue of Oberlin's theorem for spherical partial sums is unknown. In this case Carleson's theorem is also an open problem.

The topics above allow some discrete analogues, related to the characters of the cyclic group:

$$\{e^{int}\}_0^{N-1},\ t \in T_N = \left\{ 2\pi\frac{K}{N},\ K = 0, 1, \ldots, N-1 \right\}$$

This is an orthonormal system relative to invariant normalized measure on T_N. Each function f on T_N has the expansion

$$f(t) = \sum_{n=0}^{N-1} \hat{f}(n) e^{int}, \qquad \hat{f}(k) = \frac{1}{N} \sum_{t \in T_N} f(t) e^{-int}.$$

Analogous with (3) one can introduce the norm in $U(T_N)$. The following is the typical example of problem arising here: at how many (asymptotically by N) "knots" ∈ T_N can one fix a function f, defined in these points, $|f| \leq 1$, to allow the extension on T_N with U_N - norm bounded by an absolute constant? And what is an optimal way to arrange these knots? I have stated also the question about discrete analogues of Menshoff theorem. The proof is given by Kashin in [10].

II.2. Here as the space of "good" functions we consider the Wiener algebra $A(T)$ of absolute convergent series (2) with norm

$$\| f \| = \sum_{K \in N} | \hat{f}(k) |.$$

It is a remarkable object of Fourier analysis, for its investigation combines analytical, metrical, probabilistic, and even number theoretical methods, see [9].

To demonstrate some effects arising in the interpolation problem let us consider the simplest situation. Let $E = \{t_k\}_1^N$ be a finite set on the circle and we wish to extend any function $f|_E$, $|f| \leq 1$, to the circle with $\|f\|_{A(T)}$ bounded by a constant not depending on N. Is it possible? The answer depends essentially on the arithmetical nature of E. If the numbers $\{t_k\}$ are rationally independent, the answer is "yes". In this case, vectors $\vec{t_n} = nt_k(mod 2\pi), k = 1, 2, \ldots, N\}$ are dense on the torus $\boldsymbol{T_N}$ and exponents e^{int} with appropiate frequency give the required extension with arbitrary closeness.

On the other hand, if t_k are arranged in arithmetical progression the answer is "no". The norm of the optimal interpolating function in general has in this case the order $\sqrt{N}$. This explains that not every compact set of Lebesgue measure zero, even countable, is an interpolating set for the pair $(C(T), A(T))$ (such sets called by Helson sets). Meanwhile we known from Wik and Kaufman that a Helson set may be massive from the metrical point of view: it may have Hausdorff dimension equal 1. A clear description of Helson set is not available.

For a long time the problem of free interpolation by functions belonging $A(\boldsymbol{T})$ has remained open. Is the analogue of Menshoff's theorem true? The answer "no" was obtained in essentially different ways by Katznelson [11] and by Olevskii [15]. Our approach has, at the same time yielded "incorrectable" functions with maximal possible smoothness:

Theorem [15]. *There exists a function* f *belonging to Holder space* $H^{1/2}(\mathbf{T})$ *such that for every* $g \in A(\mathbf{T})$, $mes\{t: f(t) = g(t)\} = 0$.

I recall that if $f \in H^{1/2+\varepsilon}$, then it belongs to $A(T)$ without any corrections (S. Bernstein).

The approach in [15] is based on metrical considerations. Roughly speaking, we have shown that majority of continuous, and even $H^{1/2}$, functions *f* are incorrectable. This aspect was developed by Hruscev, Kahane and Katznelson [8]. They proved that the trajectories of the Brownian movement are incorrectable almost surely.

<u>References</u>

1. S. Agronsky, A.M. Bruckner, M. Laczkovich and D. Priess, Convexity conditions and intersections with smooth functions, Trans. Amer. Mat. Sec., 289 (1985), 659-677.

2. S. Agronsky, Intersections of continuous functions with families of smooth functions, Real Analysis Exchange, 10 (1984/85), 25-30.

3. J. Brown, Differentiable restrictions of real functions, Real Analysis Exchange 13 (1987/88), 44-45.

4. Z. Buczolich, Sets of convexity of continuous functions, Acta Math. Hung. 52 (1988), 291-303.

5. H. Cartan, Collected works, Springer, Berlin-New York, 1979.

6. V. Fjodoroff, On intersections of continuous functions with families of smooth functions, preprint MIEM (in Russian), RZMath. 1562 DEP (1988).

7. J. Hausserman, Porosity characteristics of intersection sets with the typical continuous function, Real Analysis Exchange, 9 (1983/84), 386-389.

8. S. Hruscev, J.-P. Kahane et Y. Katznelson, Mouvement brownien et séries de Fourier absolument convergent, C.R. Acad. Sc. Paris, 292 (1981), 389-391.

9. J.-P. Kahane, Séries de Fourier absolument convergent, Springer-Verlag, Berlin-New York, 1970.

10. B. Kashin, An analogue of Menshoff's "correction" theorem for discrete orthonormal systems, Mat. Zametki 46 (1989), 67-74.

11. Y. Katznelson, On a theorem of Menshoff, Proc. Amer. Math. Soc. 53 (1975), 396-398.

12. M. Laczkovich, Differentiable restrictions of continuous functions, Acta Math. Hung. 44 (1984), 355-360.

13. D. Oberlin, A. Rudin-Carleson theorem for uniformly convergent Taylor series, Mich. Math. J.. 27 (1980), 307-313.

14. A.M. Olevskii, Fourier series with respect to general orthogonal systems, Springer-Verlag, Berlin-New York, 1975.

15. A.M. Olevskii, The existence of functions with unremovable Carleman singularities, Dokl. Acad. Nauk. SSSR 238 (1978), Soviet Math. Dokl. 19 (1978), 102-106.

16. A.M. Olevskii, Modifications of functions and Fourier series, Uspehi Matem. Nauk, 40 (1975), 157-193.

17. A.M. Olevskii, Homeomorphisms of the circle, modifications of functions and Fourier series, Proceedings of the International Congress of Mathematicians, Berkeley, California, USA, 1986, vol. II (1987), 976-989.

18. Z. Zahorski, Sur l'ensemble des points singulière d'une fonction d'une variable réelle admettant les dérivées de tous les ordres, Fund. Math. 34 (1947), 183-245.

19. S. Ulam, A collection of mathematical problems, Interscience, New York, 1960.

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