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THE CONSTRUCTION OF A LEBESGUE MEASURABLE SET WITH EVERY DENSITY

The question of the existence of a Lebesgue measurable set $E \subseteq \mathbf{R}$ such that each density $t \in [0,1]$ occurs, was posed by R.M. Shortt. The following is the construction of such a set E.

Definition. Given a Lebesgue measurable set $E \subseteq \mathbf{R}$ and $t \in [0,1]$, $x \in \mathbf{R}$ is said to have density t with respect to E, denoted $d_E(x) = t$, if given $\epsilon > 0$ there is a $\delta > 0$ such that for all intervals I containing x with $\lambda I < \delta$,

$$\left| \frac{\lambda(I \cap E)}{\lambda I} - t \right| < \epsilon.$$

Theorem (Lebesgue Density Theorem) [1]. Given a Lebesgue measurable set $E \subseteq \mathbf{R}$, almost every point in $\mathbf{R}$ has density 0 or 1 with respect to E.

So the set of points $x \in \mathbf{R}$ where $d_E(x) \in (0,1)$ is a set of measure zero. In the following construction, for each $t \in (0,1)$ there will be an $x \in K$, the Cantor set, such that $d_E(x) = t$.

Proposition 1. Given $0 \leq \alpha \leq 1$, $\epsilon > 0$, and (a,b), there exists a measurable set $A \subseteq (a,b)$ such that $\lambda A = \alpha(b-a)$ and for every $c \in (a,b)$,

$$\left| \frac{\lambda(A \cap (a,c))}{c-a} - \alpha \right| < \epsilon \tag{1}$$

and

$$\left| \frac{\lambda(A \cap (c,b))}{b-c} - \alpha \right| < \epsilon. \tag{2}$$

Proof. Fix $n \in \mathbf{N}$. Let $m = \frac{b-a}{2}$ and put

$$A_n = \bigcup_{r=1}^{\infty} \left(a + \frac{nm}{n+r}, a + \frac{nm}{n+r} + \frac{\alpha nm}{(n+r)(n+r-1)} \right).$$

Notice that $A_n \subseteq (a, a+m]$ and that the constitutent intervals of A_n are disjoint. For any positive integer N,

$$\lambda \left(\bigcup_{r=N}^{\infty} \left(a + \frac{nm}{n+r}, a + \frac{nm}{n+r} + \frac{\alpha nm}{(n+r)(n+r-1)} \right) \right)$$

$$= \sum_{r=N}^{\infty} \left(\left(a + \frac{nm}{n+r} + \frac{\alpha nm}{(n+r)(n+r-1)} \right) - \left(a + \frac{nm}{n+r} \right) \right)$$

$$= \alpha \left(\sum_{r=N}^{\infty} \left(\frac{nm}{n+r-1} - \frac{nm}{n+r} \right) \right) = \alpha \left(\frac{nm}{n+N-1} \right).$$

In particular, for $N = 1$, $\lambda(A_n) = \alpha m$. Now take $c \in (a, a+m]$ then $c \in (a + \frac{nm}{n+s+1}, a + \frac{nm}{n+s}]$ for some integer $s \geq 0$, so

$$\frac{\alpha nm}{n+s+1} \leq \lambda(A_n \cap (a,c)) \leq \frac{\alpha nm}{n+s}$$

and

$$\frac{nm}{n+s+1} \leq (c-a) \leq \frac{nm}{n+s}.$$

Thus,

$$\frac{\alpha(n+s)}{n+s+1} \leq \frac{\lambda(A_n \cap (a,c))}{c-a} \leq \frac{\alpha(n+s+1)}{n+s}. \tag{3}$$

So for any $c \in (a, a+m]$,

$$\alpha \left(\frac{n}{n+1} \right) \leq \frac{\lambda(A_n \cap (a,c))}{c-a} \leq \alpha \left(\frac{n+1}{n} \right).$$

Take $n_0 \in \mathbf{N}$ such that $\alpha(\frac{n_0}{n_0+1}) - \alpha > -\epsilon$ and $\alpha(\frac{n_0+1}{n_0}) - \alpha < \epsilon$ then,

$$\left| \frac{\lambda(A_{n_0} \cap (a,c))}{c-a} - \alpha \right| < \epsilon. \tag{4}$$

Let A'_{n_0} be the set A_{n_0} reflected in the midpoint of (a,b). If $A = A_{n_0} \cup A'_{n_0}$ then $\lambda A = 2(\lambda A_{n_0}) = 2(\alpha m) = \alpha(b-a)$. Since A is symmetric about $\frac{b+a}{2}$ it is enough to show (1) and (2) hold for $c \in (a, a+m]$. But $\lambda(A \cap (a,c)) = \lambda(A_{n_0} \cap (a,c))$ for $c \in (a, a+m]$ so (4) implies (1). By (1)

$$\left| \alpha - \frac{\lambda(A \cap (a,c))}{c-a} \right| < \epsilon$$

which implies

$$\left| \frac{\alpha(c-b)}{c-a} + \left(\frac{\alpha(b-a)}{c-a} - \frac{\lambda(A \cap (a,c))}{c-a} \right) \right| < \epsilon$$

and since $\dfrac{\alpha(b-a)}{c-a} = \dfrac{\lambda A}{c-a} = \dfrac{\lambda(A \cap (a,c)) + \lambda(A \cap (c,b))}{c-a}$,

$$\left| \frac{\alpha(c-b)}{c-a} + \frac{\lambda(A \cap (c,b))}{c-a} \right| < \epsilon.$$

But $c \in (a, a+m]$ so $(b-c) \geq (c-a)$ thus,

$$\left| -\alpha + \frac{\lambda(A \cap (c,b))}{b-c} \right| < \epsilon$$

and (2) holds. □

Remark. As a result of (3), given $\eta > 0$ there is a $\delta > 0$ such that for all $c \in (a,b)$ with $c - a < \delta$ and all $d \in (a,b)$ with $b - d < \delta$,

$$\left| \frac{\lambda(A \cap (a,c))}{c-a} - \alpha \right| < \eta \text{ and } \left| \frac{\lambda(A \cap (d,b))}{b-d} - \alpha \right| < \eta.$$

Let f be the Cantor singular function [1]. Now construct the Cantor set K in $[0,1]$ using the process of removing middle thirds. Let $I_{n_1}, I_{n_2}, \ldots, I_{n_i} = (a_{n_i}, b_{n_i}), \ldots, I_{n_{2^{n-1}}}$ be the intervals removed from $[0,1]$ at the n^{th} step. For each $n \geq 1$ and $1 \leq i \leq 2^{n-1}$ find $E_{n_i} \subseteq I_{n_i}$ using proposition 1, where E_{n_i} is the A of proposition 1, $\alpha = f(a_{n_i})$ and $\epsilon = \frac{1}{n}$. Put

$$E = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} E_{n_i}.$$

Given a set $A \subseteq [0,1]$, the complement of A in $[0,1]$ will be written A^c.

Proposition 2. Given an interval $J \subseteq [c,d] \subseteq (\cup_{n=1}^{N} \cup_{i=1}^{2^{n-1}} I_{n_i})^c$,

$$f(c) - \frac{1}{N} \leq \frac{\lambda(J \cap E)}{\lambda J} \leq f(d) + \frac{1}{N}.$$

Proof. Since the exclusion of end points will not effect the measure, assume $J = (g,h)$ for some $g, h \in [c,d]$. Since $\lambda K = 0$ and $J \subseteq [0,1]$,

$$\lambda J = \lambda(J \cap K^c)$$

and since $J \subseteq (\cup_{n=1}^{N} \cup_{i=1}^{2^{n-1}} I_{n_i})^c$,

$$= \lambda \left(J \cap \left(\bigcup_{N+1}^{\infty} \bigcup_{i=1}^{2^{n-1}} I_{n_i} \right) \right)$$

$$= \sum_{n=N+1}^{\infty} \sum_{i=1}^{2^{n-1}} \lambda(J \cap I_{n_i}). \tag{5}$$

Since $J \subseteq (\cup_{n=1}^{N} \cup_{i=1}^{2^{n-1}} I_{n_i})^c \subseteq (\cup_{n=1}^{N} \cup_{i=1}^{2^{n-1}} E_{n_i})^c$,

$$\lambda(J \cap E) = \lambda\left(J \cap \left(\bigcup_{n=N+1}^{\infty} \bigcup_{i=1}^{2^{n-1}} E_{n_i}\right)\right).$$

$$= \sum_{n=N+1}^{\infty} \sum_{i=1}^{2^{n-1}} \lambda(J \cap E_{n_i}). \tag{6}$$

If $J \cap I_{n_i} = (a_{n_i}, h)$ and $n > N$, (1) gives

$$\left| \frac{\lambda((a_{n_i}, h) \cap E_{n_i})}{h - a_{n_i}} - f(a_{n_i}) \right| < \frac{1}{n};$$

so, since f is increasing,

$$f(c) - \frac{1}{N} < f(a_{n_i}) - \frac{1}{n} < \frac{\lambda(J \cap E_{n_i})}{\lambda(J \cap I_{n_i})} < f(a_{n_i}) + \frac{1}{n} < f(d) + \frac{1}{N}. \tag{7}$$

If $J \cap I_{n_i} = (g, b_{n_i})$ and $n > N$, the same inequalities follow similarly from (2). If $J \cap I_{n_i} = I_{n_i}$ and $n > N$, then $J \cap E_{n_i} = E_{n_i}$ and $c \leq g \leq a_{n_i}$, $b_{n_i} \leq h \leq d$. So $\lambda(J \cap E_{N_i}) = \lambda E_{n_i}$, but by the construction of $E_{n_i}, \lambda E_{n_i} = f(a_{n_i})(\lambda I_{n_i})$ so (7) again holds. Consequently, for $n > N$,

$$(f(c) - \frac{1}{N})(\lambda(J \cap I_{n_i})) \leq \lambda(J \cap E_{n_i}) \leq (f(d) + \frac{1}{N})(\lambda(J \cap I_{n_i})) \tag{8}$$

in the above three situations; while in the remaining situation $J \cap I_{n_i} = \emptyset$, (8) holds trivially. Thus, summing (8) for $1 \leq i \leq 2^{n-1}$ and $n \geq N + 1$,

$$\left(f(c) - \frac{1}{N}\right)(\lambda J) \leq \lambda(J \cap E) \leq \left(f(d) + \frac{1}{N}\right)(\lambda J)$$

follows from (5) and (6). □

Claim. Given $t \in (0, 1)$ and $x \in K$ such that $f(x) = t$, then $d_E(x) = t$. Consider two cases.

Case i) x is not an end point of K (i.e. $x \neq a_{n_i}$ or b_{n_i} for any $n \in \mathrm{N}$ and $1 \leq i \leq 2^{n-1}$). So given N there is an interval (c_N, d_N) containing

x where $d_N - c_N = \frac{1}{3^N}$ and $[c_N, d_N] \subseteq (\cup_{n=1}^{N} \cup_{i=1}^{2^{n-1}} I_{n_i})^c$ and there exists positive $\delta_N \leq \frac{1}{3^N}$ such that for every interval I containing x where $\lambda I < \delta_N$, $I \subseteq (c_n, d_N)$. So by Proposition 2

$$f(c_N) - \frac{1}{N} \leq \frac{\lambda(I \cap E)}{\lambda(I)} \leq f(d_N) + \frac{1}{N}.$$

As $N \to \infty$, c_N and d_N converge to x and $\delta_N \to 0$. Thus, since f is continuous, $f(c_N) - \frac{1}{N}$ and $f(d_N) + \frac{1}{N}$ converge to $f(x)$. So given $\epsilon > 0$ there exists N such that for all intervals I containing x with $\lambda I < \delta_N$,

$$\left| \frac{\lambda(I \cap E)}{\lambda I} - f(x) \right| < \epsilon.$$

Therefore $d_E(x) = f(x)$.

Case *ii*) x is an end point of K. So $x = a_{n_i}$ for some $n \in \mathbf{N}$ and $1 \leq i \leq 2^{n-1}$ (the case when $x = b_{n_i}$ is analogous). For a given interval I containing x look at the right portion of I, $I_r = I \cap [x, \infty)$, and the left portion of I, $I_l = I \cap (-\infty, x]$. By the argument of case *i*) given $\epsilon > 0$ there exists $\delta_r > 0$ such that for all intervals I with $x \in I$ and $\lambda I_r < \delta_r$,

$$\left| \frac{\lambda(I_r \cap E)}{\lambda I_r} - f(x) \right| < \epsilon.$$

By the remark at the end of proposition 1 there exists δ_l such that for all intervals I with $x \in I$ and $\lambda I_l < \delta_l$,

$$\left| \frac{\lambda(I_l \cap E)}{\lambda I_l} - f(x) \right| < \epsilon.$$

Thus for $\delta = \min \{\delta_r, \delta_l\}$, given any interval I with $x \in I$ and $\lambda I < \delta$,

$$\left| \frac{\lambda(I \cap E)}{\lambda I} - f(x) \right| < \epsilon.$$

Therefore $d_E(x) = f(x)$.

Since E is an open set it is clear that every point $x \in E$ has density 1 and since $E \subseteq [0,1]$ every point not in $[0,1]$ has density 0. So for each $t \in [0,1]$ there is a point $x \in \mathbf{R}$ such that $d_E(x) = t$.

REFERENCE

[1] Cohn, D.L., Measure theory. Boston: Birkhäuser 1980.

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