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ADDITIVITY OF POROUS SETS

Let **P** denote the ideal of σ-porous sets and let **K** be the ideal of first category sets on the real line **R**. The aim of this note is to prove this theorem:

THEOREM. *Let $\mathcal{I}$ be arbitrary ideal on* **R** *such that* $\mathbf{P} \subseteq \mathcal{I} \subseteq \mathbf{K}$. *Then* $\mathrm{add}(\mathcal{I}) \leq \mathbf{b}$ *and* $\mathbf{d} \leq \mathrm{cof}(\mathcal{I})$.

Let us recall some definitions (see e.g. [**2**] and [**5**]): A set $A \subseteq \mathbf{R}$ is porous if for every $b \in A$,

$$p(A,b) = \limsup_{\varepsilon \to 0+} \frac{\lambda(A,(b-\varepsilon,b+\varepsilon))}{\varepsilon} > 0.$$

Here $\lambda(A,I)$ denotes the maximal length of a subinterval of the interval I which is disjoint with A. A set A is σ-porous if it is a countable union of porous sets.

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{I}_0| : \mathcal{I}_0 \subseteq \mathcal{I} \ \& \ \bigcup \mathcal{I}_0 \notin \mathcal{I}\},$$
$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{I}_0| : \mathcal{I}_0 \subseteq \mathcal{I} \ \& \ (\forall A \in \mathcal{I})(\exists B \in \mathcal{I}_0)\, A \subseteq B\}.$$

For $f, g \in {}^\omega\omega$, $f \leq^* g$ iff $(\exists n)(\forall m > n)\, f(n) \leq g(n)$.

$$\mathbf{b} = \min\{|F| : F \subseteq {}^\omega\omega \ \& \ (\forall f \in {}^\omega\omega)(\exists g \in F)\, g \not\leq^* f\},$$

$$\mathbf{d} = \min\{|F| : F \subseteq {}^\omega\omega \ \& \ (\forall f \in {}^\omega\omega)(\exists g \in F)\, f \leq^* g\}.$$

See also [**1**].

If m, n are integers then $\langle m, n) = \{k \in \omega : m \leq k < n\}$. For $s \in {}^{<\omega}2$, $[s] = \{x \in {}^\omega 2 : s \subseteq x\}$ is a basic clopen set of the Cantor space ${}^\omega 2$. Let φ be the mapping from ${}^\omega 2$ onto the interval $\langle 0,1\rangle$ defined by $\varphi(x) = \sum_{n\in\omega} x(n)2^{-n-1}$. For $s \in {}^n 2$, $I_s = \varphi([s])$ is a closed subinterval of $\langle 0,1\rangle$ of length 2^{-n}.

The proof of the Theorem is a slight modification of A. W. Miller's proof of $\mathrm{add}(\mathbf{K}) \leq \mathbf{b}$ (see [**3**]).

Let us fix a sequence of integers $k(n)$, $n \in \omega$ such that

$$k(n+1) - k(n) > n, \quad \text{for all } n \in \omega.$$

For $f \in {}^{\omega}\omega$ let us denote

$$\alpha(f) = \{x \in {}^{\omega}2 : (\forall m) x \restriction \langle k(f(m)), k(f(m)+1)) \equiv 1\}. \tag{1}$$

LEMMA 1. *If $f \in {}^{\omega}\omega$ is increasing then the set $A = \varphi(\alpha(f))$ is porous.*

PROOF: Let $b \in A$ be arbitrary, $b = \varphi(x)$ for some $x \in \alpha(f)$. We will show that $p(A, b) = 1$.

Let $m \in \omega$ be arbitrary. Put $\varepsilon = 2^m |I_{x \restriction k(f(m)+1)}| < 2^{-m}$. For $s \in {}^{k(f(m))}2$ let s^* denote the sequence $s^\frown 1^\frown \dots ^\frown 1$ of length $k(f(m)+1)$. It is obvious that $(x \restriction k(f(m)))^* = x \restriction k(f(m)+1)$ and $A \cap I_s \subseteq I_{s^*}$. Since $|I_s| \geq 2^{f(m)} |I_{s^*}| \geq 2^m |I_{s^*}|$ and the distance between any two closest distinct intervals of the form I_{s^*} is $|I_s| - |I_{s^*}| > \varepsilon$ we have $A \cap (b - \varepsilon, b + \varepsilon) \subseteq I_{x \restriction k(f(m)+1)}$. Therefore $\lambda(A, (b-\varepsilon, b+\varepsilon)) \geq (1 - 2^{-m})\varepsilon$ and so $p(A, b) = 1$.

It is easy to see that the set A is closed and even strongly symmetrically porous, see [**5**].

LEMMA 2. *Let us denote $\varphi^{-1}(\mathcal{I}) = \{A \subseteq {}^{\omega}2 : \varphi(A) \in \mathcal{I}\}$. Then* $\mathrm{add}(\mathcal{I}) = \mathrm{add}(\varphi^{-1}(\mathcal{I}))$ *and* $\mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\varphi^{-1}(\mathcal{I}))$.

PROOF: The lemma is a simple consequence of these two implications (see e.g [**4**, Lemma 2.2]):

$$\varphi(A) \subseteq B \text{ implies } A \subseteq \varphi^{-1}(B), \text{ and}$$
$$\varphi^{-1}(B) \subseteq A \text{ implies } B \subseteq \varphi(A)$$

for $A \in \varphi^{-1}(\mathcal{I})$ and $B \in \mathcal{I}$.

According to the previous two lemmas, for proving the Theorem it is enough to prove the following:

CLAIM. *Let $\mathcal{H} = \{\alpha(f) : f \in {}^\omega\omega$ is increasing$\}$ and let $\mathcal{J}$ be an arbitrary ideal on ${}^\omega 2$ such that $\mathcal{H} \subseteq \mathcal{J} \subseteq \mathbf{K}({}^\omega 2)$. Then $\mathrm{add}(\mathcal{J}) \leq \mathbf{b}$ and $\mathbf{d} \leq \mathrm{cof}(\mathcal{J})$.*

PROOF: We will find two mappings

$$\alpha : {}^\omega\omega \to \mathcal{H} \quad \text{and} \quad \beta : \mathbf{K}({}^\omega 2) \to {}^\omega\omega$$

such that

(2) $\quad \alpha(f) \subseteq A$ implies $f \leq^* \beta(A)$ for every $A \in \mathbf{K}(2)$, $f \in {}^\omega\omega$.

This will conclude the proof because if $\mathcal{F} \subseteq {}^\omega\omega$ is any family such that $\alpha(f) \subseteq A$ for every $f \in \mathcal{F}$ then the family $\mathcal{F}$ is dominated by $\beta(A)$. Therefore $\mathrm{add}(\mathcal{J}) \leq \mathbf{b}$. The proof of $\mathbf{d} \leq \mathrm{cof}(\mathcal{J})$ is similar. Let us note that anologous details are omitted in [**2**] and [**3**].

The mapping α is already defined by (1). Let us define β. Let $A \subseteq {}^\omega 2$ be an arbitrary meager set. There is a sequence $C_0 \subseteq C_1 \subseteq C_2 \ldots$ of closed nowhere dense subsets of ${}^\omega 2$ such that $A \subseteq \bigcup_{n\in\omega} C_n$. By induction define an increasing function $g \in {}^\omega\omega$ such that

(3) $\quad (\forall s \in {}^{k(g(m)+1)}2)(\exists t \in {}^{k(g(m+1))}2)\, s \subseteq t \;\&\; [t] \cap C_m = \emptyset$

for every m. Put $\beta(A)(n) = g(2n)$.

We shall verify (2). Let $f \in {}^\omega\omega$ be arbitrary and let $f \not\leq^* \beta(A)$. We can assume that f is increasing. Therefore

$$(\forall k)(\exists n > k)\, f(n) > \beta(A)(n) = g(2n).$$

Let us denote $A_m = \langle g(m), g(m+1)) \cap \mathrm{rng}(f)$. We claim that for all $k \in \omega$ there exists $m > k$ such that $A_m = \emptyset$. To see this choose $n > k+2$ such that $g(2n) < f(n)$. Then at most n sets among A_i, $i = 0, 1, \ldots, 2n-1$ are nonempty. Hence there is an $m > n-2$ such that $A_m = \emptyset$.

Using this fact together with property (3) we can inductively define an $x \in {}^\omega 2$ and an increasing sequence n_i, $i \in \omega$ such that:

(4) $\quad (\forall i)\, [x \restriction k(g(n_i+1))] \cap C_{n_i} = \emptyset$, and

(5) $\quad (\forall m)\, x \restriction \langle k(f(m)), k(f(m)+1)\,) \equiv 1.$

The condition (5) ensures that $x \in \alpha(f)$ and since the sequence C_n, $n \in \omega$ is increasing, (4) ensures that $x \notin A$. Therefore $\alpha(f) \not\subseteq A$ and (2) holds true.

This finishes the proof of the Claim and of the Theorem.

On the one hand the Theorem gives some restrictions on the values of the cardinals add($\mathbf{P}$) and cof($\mathbf{P}$). On the other hand we still cannot decide whether inequalities add($\mathbf{P}$) $> \omega_1$ and cof($\mathbf{P}$) $< 2^\omega$ are possible.

An example of a σ-ideal which fulfils assuptions of the Claim is the σ-ideal generated by closed Lebesgue measure zero sets.

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