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Cheng-Ming Lee, Department of Mathematical Sciences
University of Wisconsin-Milwaukee
Milwaukee, Wisconsin 53201

A Symmetric Approximate Perron Integral for the Coefficient Problem of Convergent Trigonometric Series

0. Introduction

The purpose of this note is to show how a monotonicity theorem established recently by Freiling and Rinne in [FR] can be used to define a symmetric approximate Perron integral which solves the coefficient problem of the convergent trigonometric series, and to raise some questions that are interesting for further investigations.

One of the problems in the theory of trigonometric series is that of suitably defining an integral which is general enough to integrate the sum of any everywhere convergent series of the form

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

and to give back the coefficients a_n, b_n in terms of the sum function. This is the so-called coefficient problem for convergent trigonometric series. The problem seems to be first considered and solved by Denjoy in [D]. It has also been solved later by Marcinkiewicz and Zygmund [MZ], James [J] (cf. also [Z], vol II, pp. 86-91), and Burkill [BJ] (cf. also [BH]).

It is well-known that a series like (1) can converge everywhere to a function f which is not Lebesgue or even Denjoy integrable on an interval. If f were integrable, one should like to say that the series

$$\frac{1}{2} a_0 x + \sum_{n=1}^{\infty} \frac{1}{n} (a_n \sin nx - b_n \cos nx) \tag{2}$$

obtained by formally integrating (1) term by term, in some sense represents the indefinite integral of f. However, the series (2) converges, in general, only almost everywhere so that its sum function is not nice enough to be an "ordinary" indefinite integral.

Two distinct methods of attacking this difficulty have been developed. (i) Define an integral in such a way that its indefinite integral need exist only at points of a set of full measure in the interval of integration. The (T)-integral of [MZ] and the SCP-integral of [BJ] are of this type. (ii) Define a second order integral so that the indefinite integral is nice and recaptures the second primitive instead of the first one. Denjoy's totalization of second symmetric derivatives (to be denoted as $T_{2,S}$-integral in our discussion) and James' P^2-integral are of this type. The reason for this to work is that the series

$$\frac{1}{4} a_0 x^2 - \sum_{n=1}^{\infty} \frac{1}{n^2} (a_n \cos nx + b_n \sin nx) \tag{3}$$

obtained by formally integrating series (1) twice, converges everywhere to a nice function.

Representations of the coefficients a_n, b_n by using the second order $T_{2,S}$-integral and P^2-integral are not in the exact ordinary Euler-Fourier forms since the definite integrals involve the second differences of the second primitives. When the first order (T)- and (SCP)-integrals are used, the exact Euler-Fourier forms are obtained. But in proving the Euler-Fourier formulas, one still has to appeal to the properties of the function defined by the twice integrated series (3). How to avoid this is what we would like to show in this note.

In section 2, a symmetric approximate Perron integral, or simply a SAP-integral, is defined. Essentially, the SAP-integral is a process which recaptures from a symmetric approximate derivative its primitive almost everywhere up to a constant. This is possible according to a recent monotonicity theorem due to Freiling and Rinne. Their result will be stated and slightly improved in section 1. The application of the SAP-integral to the coefficient problem is discussed in section 3. Certain questions will be raised in section 4.

Throughout the note, the Lebesgue measure of a set S is denoted by $|S|$; the lower symmetric approximate derivate of F at x is denoted by $\ell D_{sap}F(x)$ and its upper one by $uD_{sap}F(x)$. Then, of course, $D_{sap}F(x)$ means the symmetric approximate derivative.

1. A Fundamental Lemma.

For the development of the SAP-integral in the next section, the following monotonicity result is essential.

Fundamental Lemma. Let f be a measurable function which is finite almost everywhere on an interval I, and let A(f) denote the set of all the points in I at which the function f is approximatly continuous. Then $|I\backslash A(f)| = 0$. Furthermore, if the lower symmetric approximate derivate, $\ell D_{sap}f$, is non-negative almost everywhere in I and is greater than $-\infty$ everywhere in I, then f is non-decreasing on A(f).

It is well-known that $|I\backslash A(f)| = 0$. (For example, see page 132 in Saks [S].) The monotonicity part in the lemma is just a slight improvement of the following result, which was established very recently by Freiling and Rinne in [FR].

THEOREM A. If a finite measurable function g has a non-negative lower symmetric approximate derivate on an interval, then g is non-decreasing on A(g), the set of all the points at which g is approximately continuous.

With theorem A, the monotonicity part of the fundamental lemma is easily proved as follows.

First, note that as f is finite almost everywhere, we may assume without loss of generality that f is finite everywhere. Next, let us denote $E = \{x: x \in I \text{ and } \ell D_{sap}(x)<0\}$. Then since $|E| = 0$, there exists a G_δ-set S such that $|S| = 0$ and $S \supset E$. Then by a result of Zahorski in [Za], there exists an absolutely continuous function h such that $h'(x)$ exists everywhere, $h'(x) = +\infty$ for $x \in S$ and $0 \leq h'(x) < +\infty$ for $x \notin S$. Now, for each $\epsilon > 0$, let $g_\epsilon = f + \epsilon h$. We see that g_ϵ satisfies all the conditions in theorem A, and hence g_ϵ is non-decreasing on the set $A(g_\epsilon)$. As $A(f) = A(g_\epsilon)$ for each $\epsilon > 0$, we conclude that f is non-decreasing on A(f).

2. The SAP-integral.

Let f be a function which is finite almost everywhere on the bounded closed interval [a,b], and let B be a measurable subset of [a,b] with

$a,b \in B$ and $|B| = b - a$. A function M is called *a symmetric approximate Perron*, or simply SAP-, *major function of f on* $[a,b]$ *with basis* B if

(i) M is measurable on $[a,b]$ and approximately continuous on B;

(ii) $\ell D_{sap}M(x) \geq f(x)$ for almost all x in $[a,b]$;

(iii) $\ell D_{sap}M(x) > -\infty$ for all x in $[a,b]$;

(iv) $M(a) = 0$.

A function m is called an SAP-*minor* function of f on $[a,b]$ with basis B if $-m$ is an SAP-major function of $-f$ on $[a,b]$ with basis B.

From the definition and the fundamental lemma in the previous section we get the following basic result.

<u>**Lemma 1**</u>. If M is an SAP-major function and m an SAP-minor function of f on $[a,b]$ with basis B, then $M-m$ is nondecreasing on B, and in particular $M(b) - m(b) \geq M(a) - m(a) = 0$.

Of course, $M-m$ here may not be defined on the set $[a,b]\setminus B$. This is immaterial since $[a,b]\setminus B$ is of measure zero.

A function f is said to be *symmetric approximate Perron*, or simply SAP-, *integrable on* $[a,b]$ *with basis* B if

$$\inf M(b) = \sup m(b) \neq \pm\infty$$

where M runs over all SAP-major and m over all SAP-minor functions of f on $[a,b]$ with basis B. The number in the above equality is called the SAP-*integral* of f on $[a,b]$ with basis B, and will be denoted as

$$(\mathrm{SAP},B) \int_a^b f(x)dx.$$

Based on lemma 1, many properties of the SAP-integral can be established. However, we will not do it here. We only mention the following one, which is needed in the next section.

THEOREM 1. Let F be a measurable function finite almost everywhere, and suppose that $-\infty < \ell D_{sap}F(x) \leq u\, D_{sap}F(x) < +\infty$ for all x, and $D_{sap}F(x) \equiv f(x)$ exists for almost all x. Then, denoting by A(F) the set of all the points at which F is approximately continuous, the function f is SAP-integrable on [a,b] with basis $A(F)\cap[a,b]$ for all $a,b \in A(F)$ with $a < b$, and

$$(SAP,\ A(F)\cap[a,b]) \int_a^b f(x)dx = F(b) - F(a).$$

PROOF. By the first part of the fundamental lemma, the complement of A(F) is of measure zero. Then the function F - F(a) serves both as an SAP-major and an SAP-minor function of f on [a,b] with basis $A(F) \cap [a,b]$, and hence the theorem is proved.

3. Application to Trigonometric Series.

THEOREM 2. Suppose that the series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

converges to a finite f(x) for every x. Then there exists a 2π-periodic set B whose complement is of measure zero such that for each $u \in B$ the functions f(x), f(x)cosnx and f(x)sinnx are SAP-integrable on $[u, u+2\pi]$ with basis $B_u \equiv B\cap[u,u+2\pi]$, and furthermore we have the following Euler-Fourier formulas:

$$\pi\, a_n = (SAP,\ B_u) \int_u^{u+2\pi} f(x)\cos nx\, dx \quad \text{for } n = 0, 1, 2, 3, \cdots,$$

$$\pi b_n = (\mathrm{SAP},\ B_u) \int_u^{u+2\pi} f(x)\sin nx\, dx \qquad \text{for } n = 1, 2, 3, \cdots .$$

Proof. It follows exactly the same line as that given in [BH] for the (SCP)-integral except that we don't have to use the series (3) obtained by integrating (1) twice. Instead, we use a theorem due to Rajchman and Zygmund in 1926 (cf. [Z], page 324), which says that if a_n, $b_n \to 0$ as $n \to \infty$, and if the series (1) converges at x_0 to a finite $f(x_0)$, then the series

$$\frac{1}{2} a_0 x + \sum_{n=1}^{\infty} \frac{1}{n} (a_n \sin nx - b_n \cos nx) \tag{2}$$

converges almost everywhere to a function $F(x)$ such that $D_{sap}F(x_0) = f(x_0)$. We proceed as follows. Since the series (1) converges in a set of positive measure, a_n, $b_n \to 0$ as $n \to \infty$. Then, by the theorem of Riesz and Fischer (cf. [Z], p. 127), the periodic part of the series (2) is the Fourier series of a locally L^2-function $G(x)$. Then by the theorem of Lebesgue (cf. [Z], p. 90), the Fourier series of $G(x)$ is (C,1)-summable to $G(x)$ for almost all x. Then the theorem of Hardy (cf. [Z], p. 78) implies that the Fourier series of $G(x)$ does converge to $G(x)$ for almost all x. Thus the series (2) does converge to $F(x) \equiv G(x) + \frac{1}{2} a_0 x$ for almost all x. (As G is locally L^2, so is F.) Now, by the theorem of Rajchman and Zygmund mentioned above, we have $D_{sap}F(x) = f(x)$ for all x. Then an application of theorem 1 in the last section shows that, taking $B = A(F)$, f is SAP-integrable on $[u, u + 2\pi]$ with basis B_u and also the Euler-Fourier formula for a_0 is obtained. The conclusion for $f(x)\cos nx$ and $f(x)\sin nx$ and the other Euler-Fourier formulas can be proved in a similar manner as that in [BH] by using the theory of formal multiplication of trigonometric series, and is omitted here.

We remark that in the above proof the almost everywhere convergence to G of the Fourier series of G can be shortened if one uses the deep theorem of Carleson, which claims that the Fourier series of a (locally) L^2-function converges almost everywhere.

4. Remarks and Questions.

(A) As mentioned in the proof of theorem 2, the theory of formal multiplication of trigonometric series is needed. However, if one can obtain a reasonable integration by parts formula for the SAP-integral of the form

$$\int_a^b f(x)G(x)dx = [F(x)G(x)]\Big|_a^b - \int_a^b F(x)g(x)dx$$

(where F is a "SAP-primitive" of f, and G is an ordinary primitive of g, and g can be as nice as is needed but including functions like sinnx and cosnx), then theorem 2 can be proved without using the formal multiplication of trigonometric series. The main difficulty to obtain such an integration by parts formula seems to lie on the fact that we don't know how to find the symmetric approximate derivative of the product of two symmetric approximate differentiable functions. Anybody is welcome to solve the following:

Problem 1. Find sufficient conditions on F and G so that the formula $(GF)'(x) = G'(x)F(x) + G(x)F'(x)$ holds. Here the derivative means the symmetric approximate derivative. [Of course, if both F and G are approximately continuous at x, then the equation holds trivially. Hence the desired condition should exclude this case.] I think the problem is nontrivial even for the ordinary symmetric derivatives.

(B) A set E is called a set of uniqueness, or U-set, if every trigonometric series like (1) converging to zero outside E vanishes identically. It is well-known that a U-set must be of measure zero but not every set of measure zero is a U-set. The quest of representing the coefficients of series (1) in terms of its sum function makes sense even if the series (1) is convergent only outside a U-set. This leads us to the following conjecture, which is stronger than the theorem by Freiling and Rinne quoted in section 1.

<u>CONJECTURE</u>. If a measurable finite function has a non-negative lower symmetric approximate derivate everywhere except on a U-set, then the function is non-decreasing on the set of points at which the function is approximately continuous.

It might be hard to attack this problem since no characterization of U-sets is known. However, it is known that every countable set is a U-set. Thus, one may want to try the case when the U-set is countable.

Note that if the conjecture is true, then the SAP-integral can be modified in such a way that it solves the coefficient problem in the best possible way without using the series (3).

(C) In solving the coefficient problem, only Denjoy's $T_{2,S}$-integral is defined constructively. All the other integrals are of Perron type. To my knowledge, there are no constructive definitions for any of the integrals. The (T)-, P^2- and SCP-integrals all somehow involve the second symmetric derivative, so that a constructive definition for each of these may be as complicated as that of

Denjoy $T_{2,S}$-integral. It might be easier to obtain a constructive definition for the SAP-integral, since it involves only the first order symmetric approximate derivative. As a first step we formulate the following generalization of the problem of how to construct an ordinary primitive.

Problem 2. Knowing the symmetric approximate derivative of a continuous function, design a constructive process to recapture the continuous function up to a constant.

(D) Thanks be given to G. Cross, who has pointed out to the author that Preisso and Thomson have done an extensive work on the approximate symmetric integral in [PT]. The SAP-integral is just their $\mathcal{A}$-Perron integral. They have proved the fundamental lemma independently of the work in [FR]. Their work is given in a very general setting based essentially on the concept of the theory of Henstock-Kurzweil integral. We hope that our simple presentation here does serve some purposes for people who are interested mainly in integrals of Perron type.

Added in proof. The conjecture as stated in remark (B) is false. [Let f denote the characteristic function of the open interval (a,b). Then f is measurable and has non-negative symmetric derivate everywhere except at b. But f is not nondecreasing on the set of points at which the function is approximately continuous.] However we think it might hold true if "measurable" is replaced by "Darboux."

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