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BAIRE MEASURES ON $[0,\Omega)$ AND $[0,\Omega]$. II

In a preceding paper [5] we have proved elementary decomposition theorems for Baire measures on the ordinal spaces $X = [0,\Omega)$ and $\bar{X} = [0,\Omega]$, where Ω denotes the first uncountable ordinal. The purpose of this note is to show that every finite Baire or Borel measure on X or $\bar{X}$ is perfect and is complete if and only if it is not purely discontinuous. It is also shown that the Baire σ-algebras $\mathcal{B}_0(X)$ and $\mathcal{B}_0(\bar{X})$ are separated and strongly Blackwell but not countably generated and that the Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(\bar{X})$ are not countably generated nor strongly Blackwell. We shall follow the terminology and notation of [5].

1. Perfectness and Completeness

Let E be X or $\bar{X}$. It is known ([1], [5]) that the Baire σ-algebra $\mathcal{B}_0(E)$ consists of the countable subsets of X together with their complements in E and that the Borel σ-algebra $\mathcal{B}(E)$ consists of the subsets A of E such that A or E-A contains an unbounded closed subset of X. It is well known that $\mathcal{B}(E) \neq \mathcal{P}(E)$, the power set of E (see, e.g., [4, Lemma 7.6] or [8]).

A finite measure μ on a measurable space $(E,\mathcal{E})$ is called *perfect* if for each $\mathcal{E}$-measurable real-valued function f on E and for each subset A of the real line $\mathbb{R}$ with $f^{-1}(A) \in \mathcal{E}$, there exists a Borel subset B of $\mathbb{R}$, i.e. $B \in \mathcal{B}(\mathbb{R})$, such that $B \subset A$ and $\mu(f^{-1}(B)) = \mu(f^{-1}(A))$ or, equivalently, for each $\mathcal{E}$-measurable real-valued function f on E, there exists a $B \in \mathcal{B}(\mathbb{R})$ such that $B \subset f(E)$ and $\mu(f^{-1}(B)) = \mu(E)$ (see [7, Lemma 2]).

We need in the sequel the following lemma.

LEMMA 1. For every real-valued function f defined on X[resp. $\bar{X}$], f is Borel measurable on X [resp. $\bar{X}$] if and only if it is constant on an unbounded closed subset of X.

PROOF. Suppose f is Borel measurable on X [resp. $\bar{X}$]. Then there exist a decreasing sequence $\{[a_n,b_n)\}$ of bounded intervals in $\mathbb{R}$ and a decreasing sequence $\{F_n\}$ of unbounded closed subsets of X such that $b_n - a_n = 1/2^{n-1}$ and $F_n \subset f^{-1}([a_n,b_n))$ for all n. Since $\cap F_n$ is an unbounded closed subset of X and since $\cap [a_n,b_n) = \{c\}$ for some $c \in \mathbb{R}$, we get $\cap F_n \subset f^{-1}(c)$. Plainly the converse of the foregoing assertion holds. □

By a minor modification of the proof of Lemma 1 we obtain the next result.

LEMMA 2. For every real-valued function f defined on X [resp. $\bar{X}$], f is Baire measurable on X [resp. $\bar{X}$] if and only if there exist $\alpha \in X$ and $c \in \mathbb{R}$ such that $[\alpha,\Omega) \subset f^{-1}(c)$ [resp. $[\alpha,\Omega] \subset f^{-1}(c)$].

THEOREM 1. Every finite Baire or Borel measure on X is perfect.

PROOF. If f is a Baire measurable function on X, then $f(X)$ is a countable subset of $\mathbb{R}$ by Lemma 2. Consequently every finite Baire measure on X is perfect (cf. [7, Example 2]).

Suppose μ is a finite Borel measure on X. Then the set $Y = \{x\colon \mu(\{x\}) > 0\}$ is countable. Let δ denote the Borel measure on X such that $\delta(A) = 1$ or 0, depending on whether A or X-A contains an unbounded closed subset of X. By Theorems 2 and 3 of [5], we also have $\mu = \nu + p\delta$, where $0 \le p < \infty$ and ν is a Borel measure concentrated on Y. Let f be any Borel measurable function on X. By Lemma 1, there exist an unbounded closed subset F of X and a real number c such that $F \subset f^{-1}(c)$. Put $B = f(Y \cup F)$. It is easy to verify that B is a countable subset of f(X) such that

$$\mu(X) = \nu(Y) + p = \mu(Y \cup F) \le \mu(f^{-1}(B)) \le \mu(X) .$$

Therefore, μ is perfect. □

By definition, every Baire or Borel measure on $\bar{X}$ is a finite measure. An application of [1, Theorem], together with Lemmas 1 and 2, yields the next result.

THEOREM 2. Every Baire or Borel measure on $\bar{X}$ is perfect.

Recall that on X, every Baire or Borel measure is regular if and only if it is purely discontinuous ([5], p. 459).

THEOREM 3. Let μ be either a Baire measure or a finite Borel measure on X. Then μ is complete if and only if it is not purely discontinuous.

PROOF. Suppose μ is a regular Baire [resp. Borel] measure on X. Then there exists an $\alpha \in X$ such that $\mu([\alpha,\Omega)) = 0$. Since $\mathcal{B}(X) \ne \mathcal{P}(X)$, there is a subset A of $[\alpha,\Omega)$ that is not Borel. Therefore, μ is not complete.

Suppose μ is a nonregular finite Borel measure on X. By Theorems

2 and 3 of [5] there is a positive number p such that $\mu(A) \geq p\ \delta(A)$ for every $A \in \mathcal{B}(X)$. If $A \in \mathcal{B}(X)$, $\mu(A) = 0$, and $B \subset A$, then $\delta(A) = 0$ so that $X-B$ contains an unbounded closed subset of X and hence $B \in \mathcal{B}(X)$. Thus μ is complete.

Using Theorem 1 of [5] we show easily that every nonregular Baire measure on X is complete. □

On the set $\bar{X}$, it is known ([1], [5]) that every Baire measure is regular and that a Borel measure is regular if and only if it is purely discontinuous. The next theorem follows easily from Theorem 5 of [5] together with Theorem of [1].

THEOREM 4. Let μ be a Baire or Borel measure on $\bar{X}$. Then μ is complete if and only if it is not purely discontinuous.

2. Strongly Blackwell Spaces

Let $(E,\mathcal{E})$ be a measurable space. For each $x \in E$, the *atom* $A(x)$ of $\mathcal{E}$ determined by x is the intersection of those sets in $\mathcal{E}$ that contain x. The σ-algebra $\mathcal{E}$ is called *separated* if $A(x) = \{x\}$ for each $x \in E$. The σ-algebra $\mathcal{E}$ is called *countably generated* (c.g.) if there is a sequence $\{E_n\}$ of elements of $\mathcal{E}$ which generates $\mathcal{E}$. If the σ-algebra $\mathcal{E}$ has a countable generator $\{E_n\}$, then for each $x \in E$, the atom $A(x)$ is the intersection of those E_n or $E - E_n$ that contain x and hence $A(x) \in \mathcal{E}$. The space $(E,\mathcal{E})$, or the σ-algebra $\mathcal{E}$, is called *separable* if $\mathcal{E}$ is c.g. and separated (see, e.g., [2], [3]).

A measurable space $(E,\mathcal{E})$, or the σ-algebra $\mathcal{E}$, is called *strongly Blackwell* if any two c.g. sub-σ-algebras of $\mathcal{E}$ with the same atoms coincide. By a *strongly Blackwell separable space* we shall mean a separable measurable space that is strongly Blackwell. It is easy to verify that the theorem of Ramachandran [6] (see also [2]) characterizing strongly Blackwell separable spaces is also valid for arbitrary strongly Blackwell spaces. In particular, we have that a measurable space $(E,\mathcal{E})$ is strongly Blackwell if and only if for every $\mathcal{E}$-measurable real-valued function f on E, $\mathcal{A}_f = \mathcal{B}_f$ where $\mathcal{A}_f = \{f^{-1}(A) \in \mathcal{E}: A \subset \mathbb{R}\}$ and $\mathcal{B}_f = \{f^{-1}(B): B \in \mathcal{B}(\mathbb{R})\}$.

We begin with a preliminary result.

LEMMA 3. Let E be X or $\bar{X}$. Then the Baire and Borel σ-algebras $\mathcal{B}_0(E)$ and $\mathcal{B}(E)$ are separated but not countably generated.

PROOF. It is easy to show that $\mathcal{B}_0(E)$ and $\mathcal{B}(E)$ are separated. Suppose $\mathcal{B}(X)$ is c.g. Then there is a Borel isomorphism f from $(X,\mathcal{B}(X))$ onto a subset Z of $[0,1]$ with its Borel σ-algebra $\mathcal{B}(Z)$ (see, e.g,, [2], [3]) which is impossible by Lemma 1. Similarly $\mathcal{B}_0(X)$, $\mathcal{B}_0(\bar{X})$ and $\mathcal{B}(\bar{X})$ are not c.g. by Lemmas 2 and 1, respectively. □

LEMMA 4. Let E be X or $\bar{X}$. Then every c.g. sub-σ-algebra of $\mathcal{B}_0(E)$ can have only a countable number of atoms and is not separated.

PROOF. Suppose $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{B}_0(X)$ with a countable generator $\{C_n\}$. We may assume without loss of generality that all C_n are countable subsets of X. Then for each $x \in X$, the atom $A(x)$ of $\mathcal{C}$ determined by x is in $\mathcal{C}$. If we put $C = \cup C_n$, then $X - C \neq \phi$ and $A(x) = X - C$ for all x in $X-C$. Consequently the set $X-C$ is an unbounded atom of $\mathcal{C}$ and hence $\mathcal{C}$ is not separated. Since the set C is countable and since $C = \cup \{A(x): x \in C\}$, there exists a countable collection $\{A(x_n): x_n \in C\}$ of bounded atoms such that $C = \cup A(x_n)$. Therefore, the family of atoms of $\mathcal{C}$ consists of X-C, together with $\{A(x_n)\}$. Similarly we also prove the lemma for $\mathcal{B}_0(\bar{X})$. □

THEOREM 5. The Baire σ-algebras $\mathcal{B}_0(X)$ and $\mathcal{B}_0(\bar{X})$ are strongly Blackwell.

PROOF. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are c.g. sub-σ-algebras of $\mathcal{B}_0(X)$ with the same atoms. Let A be any nonempty set in $\mathcal{C}$. By Lemma 4, together with hypothesis, the set A is the union of a countable collection of atoms of $\mathcal{C}$ or, equivalently, $\mathcal{D}$. Consequently $A \in \mathcal{D}$ and hence $\mathcal{C} \subset \mathcal{D}$. Similarly we get $\mathcal{D} \subset \mathcal{C}$. Thus $\mathcal{B}_0(X)$ is strongly Blackwell. Again using Lemma 4 we show that $\mathcal{B}_0(\bar{X})$ is strongly Blackwell. □

THEOREM 6. The Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(\bar{X})$ are not strongly Blackwell.

PROOF. First, we show $\mathcal{B}(X)$ is not strongly Blackwell. Let F be the set of limit ordinals in X and $G = X - F$. Then F is an unbounded closed set and G is an unbounded open set. Let $f: X \to [0,1]$ be such that $f(F) = \{1\}$ and the restriction of f to G is a bijection between G and $[0,1)$. By Lemma 1, f is Borel measurable. Let P be a subset of $[0,1)$ which is not Lebesgue measurable and $Q = [0,1] - P$. Since $F \subset f^{-1}(Q)$, we have $f^{-1}(Q) \in \mathcal{A}_f$. If $f^{-1}(Q) \in \mathcal{B}_f$, then we have $f^{-1}(Q) = f^{-1}(S)$ for some Borel subset S of $[0,1]$, whence $Q = S$, a contradiction. Consequently $\mathcal{B}_f \neq \mathcal{A}_f$ and hence $\mathcal{B}(X)$ is not strongly Blackwell. Similarly we also show that $\mathcal{B}(\bar{X})$ is not strongly Blackwell.□

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