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S-NULL FUNCTIONS

1 Introduction

In this paper we consider the class of S-null functions, i.e. of those real functions which have symmetric variation equal to zero. We prove that any S-null function is constant on a co-countable set and belongs to the first class of Baire. The results of the paper extend some theorems on locally symmetric functions to the class of S-null functions.

Let $\delta : R \to (0,\infty)$ be a function. A collection $P = \{([x_i - h_i, x_i + h_i], x_i) : i = 1,2,\ldots,n\}$ is called a symmetric δ-partition on R if $0 < h_i < \delta(x_i)$ and $(x_i - h_i, x_i + h_i) \cap (x_j - h_j, x_j + h_j) = \emptyset$, $i \neq j$. The closed interval $[a,b]$ has a symmetric δ-partition if there exists a symmetric δ-partition P on R with $\bigcup_{i=1}^{n}[x_i - h_i, x_i + h_i] = [a,b]$.

Definition 1.1. Let $f : R \to R$ and $\delta : R \to (0,\infty)$ be real functions. Define the number $V(f,\delta)$ as follows: $V(f,\delta) = \sup\{\sum_{i=1}^{n} |f(x_i + h_i) - f(x_i - h_i)| : P = \{([x_i - h_i, x_i + h_i], x_i);\ i = 1,2,\ldots,n\}$ is a symmetric δ-partition on $R\}$. The symmetric variation of f on R is defined by $VS(f) = \inf\{V(f,\delta) | \delta : R \to (0,\infty)\}$.

A function $f : R \to R$ with $VS(f) = 0$ is called an S-null function. The class of these functions is denoted by $V(S)$.

Definition 1.2. We call a function $f : R \to R$ locally symmetric if for each $x \in R$ there exists $\delta(x) > 0$ with $f(x-h) = f(x+h)$ whenever $0 < h < \delta(x)$.

I.Z. Ruzsa [1] proved the following theorem:

Theorem 1.3. If f is a locally symmetric function, then there exists $\alpha \in R$ for which the closure of the set $\{x \in R : f(x) \neq \alpha\}$ is countable.

Every locally symmetric function is an S-null function; therefore one can ask whether a similar theorem is valid for the class $V(S)$.

2 The class $V(S)$

To obtain the main result (some version of Theorem 1.3 for the class $V(S)$), we need a few lemmas.

Lemma 2.1 ([2]) Let $[a,b] \supset R$ and $\delta : R \to (0,\infty)$ be a function. Write $c = (a+b)/2$. Then there exists a set $D \subset (c,b)$ such that the closure of the set $(c,b)\backslash D$ is countable and moreover the interval $[c-x, c+x]$ has at least one symmetric δ-partition for all $x \in \left(0, \frac{b-a}{2}\right)$ with $c+x \in D$.

From this lemma we immediately deduce the following:

Lemma 2.2 ([3]). Let $\delta : R \to (0,\infty)$ be a function. Then there exists a countable set N such that for all $x, y \in R\backslash N$ the closed interval $[x,y]$ has at least one symmetric δ-partition.

Theorem 2.3. Let $f : R \to R$ be an S-null function. Then there exists $\alpha \in R$ such that the set $\{x \in R : f(x) \neq \alpha\}$ is countable and moreover for each $\varepsilon > 0$ the closure of the set $\{x \in R : |f(x) - \alpha| > \varepsilon\}$ is countable.

Proof. First we prove that f is constant on a co-countable set. To each n there corresponds a positive function $\delta_n : R \to (0,\infty)$ with $V(f,\delta_n) < \frac{1}{n}$. Lemma 2.2 implies that there is a countable set N_n such that for every $x, y \in R\backslash N_n$ the closed interval $[x,y]$ has a symmetric δ_n-partition. Put $N_0 = \bigcup_{n=1}^{\infty} N_n$. Thus N_0 is countable. Let $x, y \in R\backslash N_0$. Then for each n there exists at least one symmetric δ_n-partition of $[x,y]$, $P_n = \{([x_i - h_i, x_i + h_i], x_i) : i = 1,2,\dots,k\}$. We have that $|f(x) - f(y)| = |\sum_{i=1}^{k} f(x_i + h_i) - f(x_i - h_i)| \leq V(f,\delta_n) < \frac{1}{n}$ for all n. Consequently $f(x) = f(y)$. Since $x, y \in R\backslash N_0$ were chosen arbitrary, it follows that f is equal to a constant (say α) on $R\backslash N_0$.

We prove now the second assertion. Suppose the contrary for some $\varepsilon > 0$ the set $\overline{E_\varepsilon = \{x \in R : |f(x) - \alpha| > \varepsilon\}}$ is uncountable. Then the set of condensation points of E_ε is nonempty. Let x_0 belonging to it. Without loss of generality, we assume that $(x_0 - 1, x_0) \cap E_\varepsilon$ is uncountable. It is easy to prove that the set $\{(x+y)/2 : x, y \in S\}$ is countable if S is. Thus with $S = \{x \in R : f(x) \neq \alpha\}$ we obtain that the set $\{(x+y)/2 : f(x), f(y) \neq \alpha\}$ is countable, and we may find a point $c < x_0 - 1$ not belonging to it. For $\varepsilon > 0$ and $\delta : R \to (0,\infty)$ such that $V(f,\delta) < \varepsilon$ we use Lemma 2.1 with $a = 2c - x_0$ and $b = x_0$ to find a set D with the properties described there. Let $x \in (x_0 - 1, x_0)$, $f(x) \neq \alpha$. We have $f(2c - x) = \alpha$. If $x \in D$, then the interval $[2c - x, x]$ has at least one symmetric δ-partition. Consequently $|f(x) - f(2c - x)| = |f(x) - \alpha| \leq \sum_{i=1}^{n} |f(x_i + h_i) - f(x_i - h_i)| < V(f,\delta) < \varepsilon$. Then $x \in (x_0 - 1, x_0)$ and $|f(x) - \alpha| > \varepsilon$

imply that x belongs to $(c, x_0)\setminus D$. From Lemma 2.1 it follows that $(x_0-1, x_0)\cap E_\epsilon$ is countable. This contradiction shows that E_ϵ has no point of condensation. Consequently E_ϵ is countable for all $\varepsilon > 0$. The proof is complete.

The following theorem generalizes the well-known theorem of Kostyrko, Neubrunn, Smital and Šalát stating that every locally symmetric function is of class Baire-one [4].

Theorem 2.4. Each S-null function is of class Baire-one.

Proof. If f is not Baire 1, then there must exist a perfect set P such that the restriction of f to P has no point of continuity. Let $x \in P \cap \{x \in R : f(x) = \alpha\}$. Then exists n_0 such that $x \in E_{\frac{1}{n_0}}$. Consequently $P \subset \bigcup_{n_0=1}^{\infty} E_{\frac{1}{n_0}}$ and thus the set P is countable. This contradiction shows that f is of class Baire-one.

References

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