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ANOTHER APPROACH TO THE CONTROLLED CONVERGENCE THEOREM

The controlled convergence theorem is a convergence theorem for the Henstock integral. See Lee and Chew [1,2] for one proof of this theorem. The approach taken there is to actually find the gauge function δ. In this paper, we present a proof that uses the descriptive characterization of the Henstock integral. A function f is Henstock integrable on $[a,b]$ if and only if there exists an ACG_* function F on $[a,b]$ such that $F' = f$ almost everywhere on $[a,b]$.

We will assume that the reader is familiar with ACG and ACG_* functions (see Saks [4]), as well as the Henstock integral. Let $\mathcal{P} = \{(x_i,[c_i,d_i]) : 1 \le i \le N\}$ be a finite collection of non-overlapping tagged intervals in $[a,b]$. We will always assume that the tag is a point in the interval. Let δ be a positive function defined on $[a,b]$. We say that $\mathcal{P}$ is subordinate to δ if $[c_i,d_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ for each i. Let $\overline{E}$ be the closure of the set E and $\omega(F,[c,d])$ be the oscillation of the function F on the interval $[c,d]$.

DEFINITION: Let $\{F_n\}$ be a sequence of ACG functions defined on $[a,b]$ and let $E \subset [a,b]$. The sequence $\{F_n\}$ is equi-uniformly ACG (ACG_*) on E if E can be written as a countable union of sets on each of which the sequence $\{F_n\}$ is equi AC (AC_*).

Here is a brief explanation of the term equi-uniformly ACG. A family $\{F_\alpha\}$ of ACG functions on E is uniformly ACG on E if E can be written as a countable union of sets on each of which each F_α is AC. The term equi-uniformly ACG then indicates that not only is there a common decomposition, but that the functions are equi AC on each set. We note that the sequence $\{F_n\}$ of continuous functions is equi AC_* on $\overline{E}$ if it is equi AC_* on E.

We begin with several lemmas. The proof of the first is a routine exercise.

LEMMA 1: Let $\{F_n\}$ be a sequence of functions defined on $[a,b]$ and suppose that $\{F_n\}$ converges pointwise to a continuous function F on $[a,b]$. If $\{F_n\}$ is equi-uniformly ACG_* on $[a,b]$, then F is ACG_* on $[a,b]$.

Let $F : [a,b] \to R$ and let E be a subset of $[a,b]$. Define

$$V(F,E) = \sup\Big\{\sum_{i=1}^{n} |F(d_i) - F(c_i)|\Big\} \quad \text{and} \quad V_*(F,E) = \sup\Big\{\sum_{i=1}^{n} \omega(F,[c_i,d_i])\Big\}$$

where the supremum is taken over all finite collections $\{[c_i, d_i]\}$ of non-overlapping intervals whose endpoints belong to E. The following version of the Vitali convergence theorem is needed in the proof of the next lemma. See Natanson [3].

VITALI CONVERGENCE THEOREM: Let $\{f_n\}$ be a sequence of Lebesgue integrable functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges to f almost everywhere on $[a,b]$. If the sequence $\{\int_a^x f_n\}$ is equi AC on $[a,b]$, then f is Lebesgue integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

LEMMA 2: Let $\{f_n\}$ be a sequence of Lebesgue integrable functions defined on $[a,b]$ and let $F_n(x) = \int_a^x f_n$ for each n. Suppose that $\{f_n\}$ converges to f almost everywhere on $[a,b]$ and that $\{F_n\}$ converges pointwise to 0 on $[a,b]$. If the sequence $\{F_n\}$ is equi AC on $[a,b]$, then the sequence $\{V(F_n,[a,b])\}$ converges to 0.

PROOF: By the Vitali Convergence Theorem, the function f is Lebesgue integrable on $[a,b]$ and $\int_a^x f = \lim_{n\to\infty} F_n(x) = 0$ for all $x \in [a,b]$. Hence $f = 0$ almost everywhere on $[a,b]$. Now $\{|f_n|\}$ converges to 0 almost everywhere on $[a,b]$ and the sequence $\{\int_a^x |f_n|\}$ is equi AC on $[a,b]$. Applying the Vitali Convergence Theorem once again, we find that $\lim_{n\to\infty} V(F_n,[a,b]) = \lim_{n\to\infty} \int_a^b |f_n| = 0$.

LEMMA 3: Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$, let $F_n(x) = \int_a^x f_n$ for each n, and let E be a closed subset of $[a,b]$. Suppose that $\{f_n\}$ converges to f almost everywhere on $[a,b]$ and that $\{F_n\}$ converges uniformly to 0 on $[a,b]$. If the sequence $\{F_n\}$ is equi AC_* on E, then the sequence $\{V_*(F_n,E)\}$ converges to 0.

PROOF: Without loss of generality, we may assume that $a, b \in E$. Let $[a,b] - E = \bigcup_k (a_k, b_k)$. For each k, let $u_k = a_k + 0.3(b_k - a_k)$ and $v_k = a_k + 0.7(b_k - a_k)$. The set $A = E \cup \{u_k\} \cup \{v_k\}$ is closed. For each n, define a function G_n on $[a,b]$ by setting

$$G_n(x) = \begin{cases} F_n(x), & \text{if } x \in E; \\ \inf\{F_n(x) : x \in [a_k,b_k]\}, & \text{if } x = u_k; \\ \sup\{F_n(x) : x \in [a_k,b_k]\}, & \text{if } x = v_k; \end{cases}$$

for $x \in A$ and letting G_n be linear on the intervals contiguous to A. Note that $\omega(F_n,[c,d]) = \omega(G_n,[c,d])$ for each interval $[c,d] \subset [a,b]$ with endpoints in E. We will prove that the sequence $\{G_n\}$ is equi AC_* on A.

Let $\epsilon > 0$. Choose $\eta_1 > 0$ such that $\sum_i \omega(F_n,[c_i,d_i]) < \epsilon/4$ for all n whenever $\{[c_i,d_i]\}$ is a finite collection of non-overlapping intervals whose endpoints belong to E and satisfy $\sum_i (d_i - c_i) < \eta_1$. Choose a positive integer M such that $\sum_M^\infty (b_k - a_k) < \eta_1/2$ and let $\eta = \min\{\eta_1/2, \{0.2(b_k - a_k) : 1 \le k \le M\}\}$. Suppose that $\{[c_i,d_i] : 1 \le i \le p\}$ is a finite collection of non-overlapping intervals whose endpoints belong to A and satisfy $\sum_1^p(d_i - c_i) < \eta$. By subdividing some of these intervals if necessary, we may assume that either $E \cap \{c_i, d_i\} \neq \emptyset$ or $[c_i, d_i] \subset (a_k, b_k)$ for some k. Let $\pi_b = \{i : c_i, d_i \in E\}$, $\pi_l = \{i : c_i \in E, d_i \notin E\}$, $\pi_r = \{i : c_i \notin E, d_i \in E\}$, and $\pi_0 = \{i : c_i. d_i \notin E\}$. Fix n. We first observe that

$$\sum_{i\in\pi_b} \omega(G_n, [c_i, d_i]) = \sum_{i\in\pi_b} \omega(F_n, [c_i, d_i]) < \epsilon/4.$$

For each $i \in \pi_0$, there exists a unique $k_i \geq M$ such that $(c_i, d_i) \subset (a_{k_i}, b_{k_i})$. Hence

$$\sum_{i\in\pi_0} \omega(G_n, [c_i, d_i]) \leq \sum_{i\in\pi_0} \omega(G_n, [a_{k_i}, b_{k_i}]) = \sum_{i\in\pi_0} \omega(F_n, [a_{k_i}, b_{k_i}]) < \epsilon/4.$$

For each $i \in \pi_r$, there exists a unique $k_i \geq M$ such that $a_{k_i} < c_i < b_{k_i}$. We then have

$$\begin{aligned}\sum_{i\in\pi_r} \omega(G_n, [c_i, d_i]) &\leq \sum_{i\in\pi_r} \Big(\omega(G_n, [a_{k_i}, b_{k_i}]) + \omega(G_n, [b_{k_i}, d_i])\Big) \\ &= \sum_{i\in\pi_r} \Big(\omega(F_n, [a_{k_i}, b_{k_i}]) + \omega(F_n, [b_{k_i}, d_i])\Big) \\ &< \epsilon/4.\end{aligned}$$

The same result holds for the sum over π_l. Combining all of these inequalities, we find that $\sum_i \omega(G_n, [c_i, d_i]) < \epsilon$. This shows that the sequence $\{G_n\}$ is equi AC_* on A.

Now the sequence $\{G_n\}$ is equi AC on $[a, b]$ and converges pointwise to 0 on $[a, b]$. Each of the functions G'_n is Lebesgue integrable on $[a, b]$ and $G_n(x) = \int_a^x G'_n$ for each n. Furthermore, the sequence $\{G'_n\}$ converges to 0 on $[a, b] - A$ and converges to f almost everywhere on A. Let $\epsilon > 0$. By the previous lemma, there exists an integer N such that $\{V(G_n, [a, b])\} < \epsilon$ for all $n \geq N$. Suppose that $n \geq N$ and let $\{[c_i, d_i]\}$ be a finite collection of non-overlapping intervals whose endpoints belong to E. Then

$$\sum_i \omega(F_n, [c_i, d_i]) = \sum_i \omega(G_n, [c_i, d_i]) \leq V(G_n, [a, b]) < \epsilon$$

and it follows that $V_*(F_n, E) \leq \epsilon$. This completes the proof.

LEMMA 4: Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a, b]$ and let $F_n(x) = \int_a^x f_n$ for each n. Suppose that $\{f_n\}$ converges to f almost everywhere on $[a, b]$ and that the sequence $\{F_n\}$ is equi-uniformly ACG_* on $[a, b]$. Let $[a, b] = \bigcup_i E_i$ where each E_i is closed and $\{F_n\}$ is equi AC_* on each E_i. If $\{F_n\}$ converges uniformly to F on $[a, b]$, then $\{V_*(F_n - F, E_i)\}_{n=1}^{\infty}$ converges to 0 for each i.

PROOF: The function F is ACG_* on $[a, b]$ by Lemma 1. It follows that F' exists almost everywhere on $[a, b]$ and is Henstock integrable on $[a, b]$. Fix i. The sequences $\{f_n - F'\}$ and $\{F_n - F\}$ satisfy all of the hypotheses of Lemma 3 on E_i. Hence $\{V_*(F_n - F, E_i)\}$ converges to 0.

LEMMA 5: Let $G : [a,b] \to R$ and let $E \subset [a,b]$ be closed. Let c and d be the bounds of E and let $\mathcal{P} = \{(x_k, [c_k, d_k]) : 1 \le k \le p\}$ be a collection of non-overlapping tagged intervals in $[c,d]$. If $x_k \in E$ for each k, then $\sum_{k=1}^{p} \omega(G, [c_k, d_k]) \le 3\, V_*(G, E)$.

PROOF: Let $[c,d] - E = \bigcup_i (a_i, b_i)$. We may assume that each of the tags of $\mathcal{P}$ occurs as an endpoint. Let $\pi_b = \{k : c_k, d_k \in E\}$, $\pi_l = \{k : d_k \notin E\}$, and $\pi_r = \{k : c_k \notin E\}$. Clearly $\sum_{k \in \pi_b} \omega(G, [c_k, d_k]) \le V_*(G, E)$. For each $k \in \pi_l$, there exists a unique integer i_k such that $a_{i_k} < d_k < b_{i_k}$. Hence

$$\sum_{k \in \pi_l} \omega(G, [c_k, d_k]) \le \sum_{k \in \pi_l} \Big(\omega(G, [c_k, a_{i_k}]) + \omega(G, [a_{i_k}, b_{i_k}]) \Big) \le V_*(G, E).$$

A similar result holds for π_r and the lemma follows.

CONTROLLED CONVERGENCE THEOREM: Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$ and let $F_n(x) = \int_a^x f_n$ for each n. Suppose that $\{f_n\}$ converges to f almost everywhere on $[a,b]$ and that $\{F_n\}$ converges uniformly to F on $[a,b]$. If the sequence $\{F_n\}$ is equi-uniformly ACG_* on $[a,b]$, then f is Henstock integrable on $[a,b]$ and $\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$.

PROOF: Let $[a,b] = \bigcup_i E_i$ where each E_i is closed and $\{F_n\}$ is equi AC_* on each E_i. Since the function F is ACG_* on $[a,b]$ by Lemma 1, it is sufficient to prove that $F' = f$ almost everywhere on $[a,b]$. We will prove that the set $D = \{x \in [a,b) : \overline{F}^+(x) \ne f(x)\}$ has measure zero. The proof for the other three Dini derivates is quite similar. Suppose that $\mu(D) > 0$. For each positive integer n, let

$$D_n = \Big\{ x \in D : \limsup_{t \to x^+} \Big| \frac{F(t) - F(x)}{t - x} - f(x) \Big| > \frac{1}{n} \Big\}.$$

Since $D = \bigcup_n \bigcup_i (D_n \cap E_i)$, there exist integers p and j such that $\mu(D_p \cap E_j) = 2\beta > 0$. Let c and d be the bounds of E_j. By Egorov's Theorem, there exists a set $B \subset D_p \cap E_j \cap (c,d)$ such that $\mu(B) > \beta$ and $\{f_n\}$ converges to f uniformly on B. Choose a positive integer N_1 such that $|f_n(x) - f(x)| < 1/(36p)$ for all $n \ge N_1$ and all $x \in B$. By Lemma 4, there exists an integer $N \ge N_1$ such that $V_*(F_n - F, E_j) \le \beta/(36p)$ for all $n \ge N$. Let δ_1 be a gauge function for f_N corresponding to $\beta/(24p)$. Let O be an open set such that $B \subset O \subset (c,d)$ and $\mu(O) < 3\beta$. For each $x \in B$, let $\delta(x)$ be the minimum of $\delta_1(x)$ and the distance from x to $[a,b] - O$.

For each $x \in B$ and for each $h > 0$, there exists $v_h^x \in (x, x+h)$ such that

$$\Big| \frac{F(v_h^x) - F(x)}{v_h^x - x} - f(x) \Big| \ge \frac{1}{p}.$$

The collection $\mathcal{I} = \bigcup_{x \in B} \{[x, v_h^x] : 0 < h < \delta(x)\}$ is a Vitali cover of B. By the Vitali Covering Lemma, there exists a finite collection $\{[c_k, d_k] : 1 \leq k \leq K\}$ of disjoint intervals in $\mathcal{I}$ such that $\sum_1^K (d_k - c_k) \geq \beta/2$. Note that each of the intervals $[c_k, d_k] \subset O \subset (c, d)$ and that $(c_k, [c_k, d_k])$ is subordinate to δ for each k. Using Henstock's Lemma and Lemma 5, we obtain

$$\begin{aligned}
\frac{\beta}{2p} &\leq \sum_{k=1}^{K} \left| F(d_k) - F(c_k) - f(c_k)(d_k - c_k) \right| \\
&\leq \sum_{k=1}^{K} \left| F(d_k) - F(c_k) - \left(F_N(d_k) - F_N(c_k)\right) \right| + \sum_{k=1}^{K} \left| F_N(d_k) - F_N(c_k) - f_N(c_k)(d_k - c_k) \right| \\
&\quad + \sum_{k=1}^{K} \left| f_N(c_k) - f(c_k) \right| (d_k - c_k) \\
&< \sum_{k=1}^{K} \omega(F - F_N, [c_k, d_k]) + \frac{2\beta}{24p} + \frac{1}{36p} \sum_{k=1}^{K} (d_k - c_k) \\
&\leq 3\, V_*(F_N - F, E_j) + \frac{\beta}{12p} + \frac{3\beta}{36p} \\
&< \frac{3\beta}{36p} + \frac{\beta}{12p} + \frac{\beta}{12p} = \frac{\beta}{4p},
\end{aligned}$$

a contradiction. We conclude that $\mu(D) = 0$.

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