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ON DECOMPOSITIONS OF QUASICONTINUITY

There are many papers which deal with decompositions of continuity (see for example [2], [5], [6], [10]). The purpose of this note is to investigate similar questions for the quasicontinuity. A characterization of the cliquishness on Baire spaces is given.

In what follows X, Y denote topological spaces. For a subset A of a topological space denote $\mathrm{Cl}\, A$ and $\mathrm{Int}\, A$ the closure and the interior of A, respectively. The letters N, Q and R stand for the set of natural, rational and real numbers, respectively.

We recall that a function $f:X \to Y$ is almost continuous (also nearly continuous) at a point $x \in X$ (see [7]) if for each neighbourhood V of $f(x)$, the set $\mathrm{Cl}\, f^{-1}(V)$ is a neighbourhood of x. Denote by H_f the set of all such points at which f is almost continuous. If $H_f = X$, then f is said to be almost continuous.

A function $f:X \to Y$ is quasicontinuous at a point $x \in X$ (see [9]) if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Denote by Q_f the set of all points at which f is quasicontinuous. If $Q_f = X$, then f is said to be quasicontinuous.

A function $f:X \to Y$ is simply continuous (see [1]) if for each open set V in Y, the set $f^{-1}(V)$ is a union of an open set and a nowhere dense set in X.

It is easy to see that every quasicontinuous function is simply continuous.

Let Y be a metric space with a metric d. A function $f:X \to Y$ is cliquish at a point $x \in X$ (see [9]) if for each $\varepsilon > 0$ and each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $d(f(y), f(z)) < \varepsilon$ for each $y, z \in G$. Denote by A_f the set of all points at which f is cliquish. If $A_f = X$, then f is said to be cliquish.

The set A_f is closed in X (see [8]). Hence, if Y is a metric space and $f:X \to Y$ is a function such that Q_f is dense in X, then f is cliquish.

Now we shall give a simultaneous generalization of the almost continuity and of the quasicontinuity.

Definition 1. We say that $f:X \to Y$ is almost quasi-continuous at a point $x \in X$, if for each neighbourhood V of $f(x)$ and each neighbourhood U of x, the set $f^{-1}(V) \cap U$ is not nowhere dense. Denote by B_f the set of all points at which f is almost quasicontinuous. If $B_f = X$, we say that f is almost quasicontinuous.

Remark 1. It is easy to see that $H_f \cup Q_f \subset B_f$.

Remark 2. Evidently, a function f is almost quasicontinuous at x if and only if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there is a nonempty open set $G \subset U$ such that $G \subset \mathrm{Cl}\, f^{-1}(V)$.

Lemma 1. Let Y be a regular space. Then $B_f \cap \mathrm{Int}\,\mathrm{Cl}\, Q_f \subset Q_f$.

Proof. Let $x \in B_f \cap \mathrm{Int}\,\mathrm{Cl}\, Q_f$. Let U and V be open neighbourhoods of x and $f(x)$, respectively. Put $H = \mathrm{Int}\,\mathrm{Cl}\, Q_f$. Choose a neighbourhood W of $f(x)$ such that $\mathrm{Cl}\, W \subset V$. From the almost quasicontinuity at x there is an open nonempty set $G \subset U \cap H$ such that $f^{-1}(W)$ is dense in G. Since $G \subset \mathrm{Cl}\, Q_f$, there is a point $y \in Q_f \cap G$. Let S be an arbitrary neighbourhood of $f(y)$. From the quasicontinuity at y there is a nonempty open set $T \subset G$ such that $f(T) \subset S$. From the density of $f^{-1}(W)$ in G we have $f^{-1}(W) \cap T \neq \emptyset$. Then $\emptyset \neq W \cap f(T) \subset W \cap S$. Thus each neighbourhood S of $f(y)$ intersects the set W, which yields $f(y) \in \mathrm{Cl}\, W \subset V$. Therefore V is a neighbourhood of $f(y)$. From the quasicontinuity at y there is a nonempty open set $E \subset U$ such that $f(E) \subset V$. Therefore $x \in Q_f$.

We recall that a set A is said to be quasiclosed (also semiclosed) if $\mathrm{Int}\,\mathrm{Cl}\, A \subset A$.

Proposition 1. Let Y be a regular space. If $f:X \to Y$ is almost quasicontinuous, then Q_f is a quasiclosed set.

From the Lemma 1 we get

Theorem 1. Let Y be a regular space. Then $f:X \to Y$ is quasicontinuous if and only if it is almost quasicontinuous and Q_f is dense set in X.

The following example shows that the assumption of the regularity of Y in Theorem 1 cannot be omitted.

Example 1. Let $X = R$ with the usual topology. Let $Y = \{a, b\}$, $\mathcal{T} = \{\emptyset, \{b\}, Y\}$. Let $f:X \to Y$, $f(x) = a$ for $x \in Q$, $f(x) = b$ otherwise. Then f is almost quasicontinuous, the set Q_f is dense in X, however f is not quasicontinuous.

Lemma 2. Let Y be a metric space with a metric d. Then $B_f \cap \operatorname{Int} A_f \subset Q_f$.

Proof. Let $x \in B_f \cap \operatorname{Int} A_f$. Let U be a neighbourhood of x and $\varepsilon > 0$. Since $x \in B_f$, there is a nonempty open set $G \subset U \cap \operatorname{Int} A_f$ such that the set $H = f^{-1}(S(f(x), \varepsilon/2))$ (where $S(f(x), \varepsilon/2) = \{w \in Y: d(f(x), w) < \varepsilon/2\}$) is dense in G. Let $y \in G \cap H$. From the cliquishness at y there is a nonempty open set $S \subset G$ such that $d(f(u), f(v)) < \varepsilon/2$ for all $u, v \in S$. Since H is dense in G, there is $z \in H \cap S$. Let $t \in S$ be an arbitrary point. Then $d(f(x), f(t)) \leq d(f(x), f(z)) + d(f(z), f(t)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $x \in Q_f$.

From Lemma 2 we get

Theorem 2. Let Y be a metric space. Then $f:X \to Y$ is quasicontinuous if and only if it is almost quasicontinuous and cliquish.

We shall give a simultaneous generalization of Theorems 1 and 2.

Definition 2. Denote $L_f = \{x \in X:$ there is a base $\mathcal{B}$ of neighbourhoods of $f(x)$ such that for each $B \in \mathcal{B}$ there is a neighbourhood U of x such that the set $f^{-1}(B) - \operatorname{Int} f^{-1}(B)$ is nowhere dense in $U\}$.

Remark 3. We observe that $X = B_f \cup L_f$.

Lemma 3. Let $f:X \to Y$ be a function. Then $\operatorname{Int} Q_f \subset L_f$.

Proof. Let $x \in \operatorname{Int} Q_f$. Let B be an open neighbourhood of $f(x)$. Put $G = \operatorname{Int} Q_f$ and $H = f^{-1}(B) - \operatorname{Int} f^{-1}(B)$. We shall show that H is nowhere dense in G. By contradiction. Let $K \subset G$ be a nonempty open set such that H is dense in K. Let $y \in H \cap K$. Then K is a neighbourhood of y and B is a neighbourhood of $f(y)$. Hence from the quasicontinuity at y there is a nonempty open set $L \subset K$ such that $f(L) \subset B$. Thus $L \subset f^{-1}(B)$, which yields $L \subset \operatorname{Int} f^{-1}(B)$. Hence $L \cap H = \emptyset$, which contradicts the density H in K.

Corollary 1. If $f:X \to Y$ is quasicontinuous, then $L_f = X$.

Remark 4. It is easy to see that if f is continuous at x, then $x \in L_f$. The following example shows that this assertion does not hold for quasicontinuity points.

Example 2. Let $f: R \to R$, $f(x) = x$ for $x \in Q$, $x \geq 0$, $f(x) = -1$ for $x \in Q$, $x < 0$ and $f(x) = 0$ otherwise. Then $0 \in Q_f - L_f$.

Lemma 4. Let Y be a regular space. Then $\text{Int Cl } Q_f \subset L_f \cup Q_f$.

Proof. According to Remark 3 and Lemma 1 we get $\text{Int Cl } Q_f = (B_f \cup L_f) \cap \text{Int Cl } Q_f = (B_f \cap \text{Int Cl } Q_f) \cup$ $\cup (L_f \cap \text{Int Cl } Q_f) \subset Q_f \cup L_f$.

Proposition 2. Let Y be a regular space. Let Q_f be a dense set in X. Then $X = L_f \cup Q_f$.

Proposition 3. If $X = L_f \cup Q_f$, then the set L_f is dense in X.

Proof. Since $X - L_f \subset Q_f$, according to Lemma 3 we have $\text{Int } (X - L_f) \subset \text{Int } Q_f \subset L_f$. On the other hand evidently $\text{Int } (X - L_f) \subset X - L_f$. Hence $\text{Int } (X - L_f) = \emptyset$, i. e. the set L_f is dense in X.

Corollary 2. Let Y be a regular space. If the set Q_f is dense in X, then the set L_f is dense in X.

Lemma 5. Let Y be a metric space. Then $\text{Int } A_f \subset L_f \cup Q_f$.

Proof. According to Remark 3 and Lemma 2 we have $\text{Int } A_f =$ $= (B_f \cup L_f) \cap \text{Int } A_f = (B_f \cap \text{Int } A_f) \cup (L_f \cap \text{Int } A_f) \subset Q_f \cup L_f$.

Proposition 4. Let Y be a metric space. Let $f:X \to Y$ be cliquish. Then $X = L_f \cup Q_f$.

From Propositions 3 and 4 we get

Corollary 3. Let Y be a metric space. Let $f:X \to Y$ be cliquish. Then the set L_f is dense in X.

Lemma 6. Let $f:X \to Y$ be a function. Then $B_f \cap L_f \subset Q_f$.

Proof. Let $x \in B_f \cap L_f$. Let U and V be neighbourhoods of x and $f(x)$, respectively. Let B be a neighbourhood of $f(x)$ such that $B \subset V$ and let T be a neighbourhood of x such that the set $H = f^{-1}(B) - \text{Int } f^{-1}(B)$ is nowhere dense in T. Since $x \in B_f$, there is a nonempty open set $G \subset U \cap T$ such that $f^{-1}(B)$ is dense in G. Since H is nowhere dense in T, there is a nonempty open set $K \subset G$ such that $H \cap K = \emptyset$. Since $f^{-1}(B)$ is dense in G, we have $f^{-1}(B) \cap K \neq \emptyset$. Since $H \cap K = \emptyset$, we get $\text{Int } f^{-1}(B) \cap K \neq \emptyset$. Put $S = \text{Int } f^{-1}(B) \cap K$. Then S is a nonempty open subset of U and $f(S) \subset V$. Therefore $x \in Q_f$.

Theorem 3. Let Y be a regular space. Then $f:X \to Y$ is quasicontinuous if and only if it is almost quasicontinuous and the set L_f is dense in X.

Proof. Necessity. According to Theorem 1 and Corollary 2. Sufficiency. According to Lemma 6 and Theorem 1.

Clearly, Theorem 3 is a generalization of Theorems 1 and 2 (by Corollary 3). Now we shall give other generalization of Theorems 1 and 2 (by Propositions 2 and 4, respectively), where the regularity of a range space is not required.

Theorem 4. Let $f:X \to Y$ be a function. Then the following three conditions are equivalent:

(i) f is quasicontinuous;

(ii) f is almost quasicontinuous and $L_f = X$;

(iii) f is almost quasicontinuous and $X = L_f \cup Q_f$.

Proof.

(i) $\Rightarrow$ (ii): according to Remark 1 and Corollary 1.

(ii) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (i): according to Lemma 6 we have $X = L_f \cup Q_f =$
$= B_f \cap (L_f \cup Q_f) \subset (B_f \cap L_f) \cup Q_f \subset Q_f$.

By the definition of the simply continuity we get

Lemma 7. Let $f:X \to Y$ be a simply continuous function. Then $L_f = X$. (The converse is not true, as the Riemann function shows.)

Theorem 5. A function $f:X \to Y$ is quasicontinuous if and only if it is almost quasicontinuous and simply continuous.

Proof. According to Lemma 7 and Theorem 4.

Now we shall give a certain characterization of the cliquishness. We recall that a topological space X has the Souslin property (see [4; p. 86]) if every family of pairwise disjoint nonempty open subsets of X is countable.

Definition 3. We say that a topological space X has the locally Souslin property if for each point of X there is its neighbourhood, which (as a subspace of X) has the Souslin property.

Example 3. Every uncountable discrete topological space has the locally Souslin property, however it has not the Souslin property.

By a routine way we can prove

Lemma 8. A topological space X is completely regular if and only if for each $a \in X$ and each neighbourhood U of a there is a family $\{B_\varepsilon\}_{\varepsilon \in (0,1]}$ of open neighbourhoods of a such that $\mathrm{Cl}\, B_\gamma \subset B_\delta \subset U$ for $0 < \gamma < \delta \leq 1$.

Theorem 6. Let a topological space X have the locally Souslin property, let Y be a completely regular space and let $f: X \to Y$ be a function. If the set Q_f is dense in X, then $L_f = X$.

Proof. Let $x \in X - L_f$. Then there is a neighbourhood W of $f(x)$ such that for each neighbourhood V of $f(x)$, $V \subset W$ and each neighbourhood T of x, the set $f^{-1}(V) - \mathrm{Int}\, f^{-1}(V)$ is not nowhere dense in T. Let U be a neighbourhood of x such that every family of pairwise disjoint nonempty open subsets of U is countable. Let $\{B_\varepsilon\}_{\varepsilon \in (0,1]}$ be a family of open neighbourhoods of $f(x)$ such that $\mathrm{Cl}\, B_\gamma \subset B_\delta \subset W$ for $0 < \gamma < \delta \leq 1$. Let $0 < \varepsilon < 1$. Then the set $H_\varepsilon =$ $= f^{-1}(B_\varepsilon) - \mathrm{Int}\, f^{-1}(B_\varepsilon)$ is not nowhere dense in U. Therefore there is a nonempty open set $G_\varepsilon \subset U$ such that H_ε is dense in G_ε. Since Q_f is dense in X, there is a point $z \in Q_f \cap G_\varepsilon$. Let S be an arbitrary neighbourhood of $f(z)$. From the quasicontinuity at z there is a nonempty open set $E \subset G_\varepsilon$ such that $f(E) \subset S$. Since H_ε is dense in G_ε, there is a point $w \in H_\varepsilon \cap E$. Then $f(w) \in S \cap B_\varepsilon$. Therefore each neighbourhood S of $f(z)$ intersects the set B_ε, i. e. $f(z) \in \mathrm{Cl}\ B_\varepsilon$. We shall show that $f(z) \notin B_\varepsilon$. By contradiction. Suppose that $f(z) \in B_\varepsilon$. From the quasicontinuity at z there is a nonempty open set $K \subset G_\varepsilon$ such that $f(K) \subset B_\varepsilon$. This yields $K \subset f^{-1}(B_\varepsilon)$ and hence also $K \subset \mathrm{Int}\ f^{-1}(B_\varepsilon)$. Since H_ε is dense in G_ε, there is a point $v \in H_\varepsilon \cap K$. Therefore $v \in H_\varepsilon \subset X - \mathrm{Int}\ f^{-1}(B_\varepsilon)$ and simultaneously $v \in K \subset \mathrm{Int}\ f^{-1}(B_\varepsilon)$, a contradiction. Therefore $f(z) \in \mathrm{Cl}\ B_\varepsilon - B_\varepsilon$. From this we get $f(Q_f \cap G_\varepsilon) \subset \mathrm{Cl}\ B_\varepsilon - B_\varepsilon$. Thus we have constructed a family $\{G_\varepsilon\}_{\varepsilon \in (0,1)}$ of nonempty open subsets of U. We shall show that $\{G_\varepsilon\}_{\varepsilon \in (0,1)}$ is a family of pairwise disjoint sets. By contradiction. Suppose that there is $0 < \gamma < \delta < 1$ such that $G = G_\gamma \cap G_\delta$ is a nonempty set. Since Q_f is dense in X, there is a point $u \in G \cap Q_f$. Then $f(u) \in \mathrm{Cl}\ B_\gamma - B_\gamma \subset$ $\subset \mathrm{Cl}\ B_\gamma \subset B_\delta$ and simultaneously $f(u) \in \mathrm{Cl}\ B_\delta - B_\delta \subset X - B_\delta$, a contradiction. From the definition of the set U it follows that $\{G_\varepsilon\}_{\varepsilon \in (0,1)}$ is a countable family and this contradicts to the uncountability of the interval $(0, 1)$. Therefore $X - L_f = \emptyset$, i. e. $X = L_f$.

The following example shows that the assumption of the locally Souslin property in Theorem 6 cannot be omitted.

Example 4. We put $T = A \times I$, where $A = \{a \in R^N: a_n \geq a_{n+1}$ for all $n \in N$ and $\lim_{n\to\infty} a_n = 0\}$ and $I = [0, 1]$. Let $S = \bigcup_{t\in T} (R \times \{t\})$ with the sum topology σ. Let $X = S \cup \{0\}$ with a topology $\tau = \sigma \cup \{X\}$. Let $Q^+ = \{q_1, q_2, q_3, \dots \}$ be the set of all positive rational numbers. For each $t = (a, r) \in T$ define a function $f_t: R \times \{t\} \to R$ as follows: $f_t(x) = r + a_n$, if $x = (q_n, t)$; $f_t(x) = r - a_n$, if $x = (-q_n, t)$ and $f_t(x) = r$ otherwise. Now we define a function $f: X \to R$ as $f(x) = f_t(x)$ for $x \in R \times \{t\}$ and $f(x) = 0$ otherwise.

Then the set $Q_f = \bigcup_{t\in T} ((R - Q) \times \{t\})$ is dense in X (and f is cliquish), however $L_f \neq X$.

We shall show that $0 \notin L_f$. Let B be an arbitrary bounded neighbourhood of the point 0 (in R). Put $r = \sup B$. We shall show that $f^{-1}(B) - \operatorname{Int} f^{-1}(B)$ is not nowhere dense in X (X is only neighbourhood of 0 in X).

a) Suppose that $r \in B$. Choose an arbitrary point $a \in A$. Put $G = (0, \infty) \times \{(a, r)\}$. Then G is a nonempty open subset of X such that $f^{-1}(B) \cap G = ((0, \infty) - Q) \times \{(a, r)\}$. This yields that $f^{-1}(B) - \operatorname{Int} f^{-1}(B)$ is dense in G.

b) Suppose that $r \notin B$. Choose $a \in A$ such that $r - a_n \in B$ for all $n \in N$. Put $G = (-\infty, 0) \times \{(a, r)\}$. Then G is a nonempty open subset of X such that $f^{-1}(B) \cap G = ((-\infty, 0) \cap Q) \times \{(a, r)\}$. This yields that $f^{-1}(B) - \operatorname{Int} f^{-1}(B)$ is dense in G.

Lemma 9. Let Y be a second countable space and let

$f: X \to Y$ be a function. Then $L_f - Q_f$ is a set of the first category.

Proof. In the paper [11] it is proved that $X - H_f$ is a set of the first category for second countable range space. Hence according to Remark 1 and Lemma 6 we have $L_f - Q_f \subset$
$\subset L_f - (L_f \cap B_f) \subset L_f - B_f \subset X - B_f \subset X - H_f$, therefore $L_f - Q_f$ is a set of the first category.

Proposition 5. Let X be a Baire space and let Y be a regular second countable space. Let $f: X \to Y$ be a function. Then the set Q_f is dense in X if and only if $X = L_f \cup Q_f$.

Proof. Necessity. According to Proposition 2.
Sufficiency. According to Lemma 9 the set $X - Q_f =$
$= (L_f \cup Q_f) - Q_f = L_f - Q_f$ is a set of the first category. Since X is a Baire space, the set Q_f is dense in X.

Corollary 4. Let X be a Baire space and Y be a separable metric space. Then $f: X \to Y$ is cliquish if and only if $X = L_f \cup Q_f$.

Proof. According to Propositions 4 and 5.

Now we shall give a new characterization of the cliquishness.

Theorem 7. Let X be a Baire space with the locally Souslin property. Let Y be a separable metric space. Then $f: X \to Y$ is a cliquish function if and only if $X = L_f$.

Proof. According to Corollary 4, Proposition 5 and

Theorem 6.

Corollary 5. A function $f:R \to R$ is cliquish if and only if $L_f = R$.

Remark 5. The assumption "$L_f = R$" in Corollary 5 cannot be replaced by the assumption "L_f is dense in R". The function $f:R \to R$, $f(x) = q$ for $x = p/q$, where p, q are relatively prime integers, $q > 0$, $f(x) = 0$ otherwise, is not cliquish, however the set L_f is dense in R.

Remark 6. There is a real function $f:X \to R$ such that f is not cliquish, however $L_f = X$. Let $X = N$ and let $\mathcal{F}$ be an ultrafilter in X, which contains no finite set. Let X be assigned the topology $\tau = \mathcal{F} \cup \{\emptyset\}$. Define $f:X \to R$ as $f(x) = x$ for all $x \in X$. Then $L_f = X$, however $A_f = \emptyset$ (see [3]).

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