Real Analysis Exchange Vol 16 (1990-91)

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A note on topologies related to (x^α)-porosity

Let (X,ρ) be a metric space. The open ball with the centre $z \in X$ and radius $r > 0$ is denoted by $B(z,r)$. Let $M \subset X$, $z \in X$ and $R > 0$. Then we denote by $\gamma(z,R,M)$ the supremum of the set of all $r > 0$ for which there exists $y \in X$ such that $B(y,r) \subset B(z,R) \setminus M$.

Let $\alpha \in (0,1]$. If

$$\limsup_{R\to 0+} \frac{\gamma(z,R,M)^\alpha}{R} > 0,$$

we say that M is (x^α)-porous at z. If $\alpha = 1$, then we simply say that M is porous at z.

Let $\alpha \in (0,1)$. If

$$\limsup_{R\to 0+} \frac{\gamma(z,R,M)^\alpha}{R} = \infty,$$

we say that M is (x^α)-strongly porous at z. If

$$\limsup_{R\to 0+} \frac{\gamma(z,R,M)}{R} \geq \frac{1}{2},$$

we say that M is (x)-strongly porous at z, or simply, strongly porous at z.

PROPOSITION 1. Let $\alpha \in (0,1]$. If z is not an isolated point of X and M is (x^{α})-porous at z ((x^{α})-strongly porous at z), then $M \cup \{z\}$ is also (x^{α})-porous at z ((x^{α})-strongly porous at z).

P r o o f. If $z \in \overline{M}$, then the inclusion $B(z_n,r_n) \subset B(z,R_n) \setminus M$ implies $B(z_n,r_n) \subset B(z,R_n) \setminus M \setminus \{x\}$. Hence the assertion holds.

If $z \notin \overline{M}$, then, for a sequence $\{z_n\}$ of points tending to z, we put $r_n = \rho(z,z_n)$ and $R_n = 2r_n$. Then, for sufficiently large n, we have $B(z_n,r_n) \subset B(z,R_n) \setminus M \setminus \{x\}$, $\frac{r_n}{R_n} = \frac{1}{2}$ and $\frac{r_n^{\alpha}}{R_n} \xrightarrow[n\to\infty]{} \infty$ for $\alpha \in (0,1)$.

In [F] the notions of (x^{α})-porosity and (x^{α},∞)-porosity ($(x,1)$-porosity for $\alpha = 1$) for subsets of the real line were investigated. Proposition 1 guarantees that if $X = R$ then those notions are equivalent to our notions of (x^{α})-porosity and (x^{α})-strong porosity.

We say that $E \subset X$ is (x^{α})-superporous at z if $E \cup F$ is (x^{α})-porous at z whenever F is (x^{α})-porous at z; E is (x^{α})-strongly superporous at z if $E \cup F$ is (x^{α})-strongly porous at z whenever F is (x^{α})-strongly porous at z. E is said to be (x^{α})-superporous ((x^{α})-strongly superporous) if it is (x^{α})--superporous ((x^{α})-strongly superporous) at all its points.

A set $G \subset X$ is said to be (x^{α})-porosity open if $X \setminus G$ is (x^{α})-superporous at any point of G. The system of all sets which are (x^{α})-superporous at a fixed point z forms an ideal. Therefore the system of all (x^{α})-porosity open sets forms a topology.

We call it the (x^{α})-porosity topology and denote by T_{α}. In the same way we define (x^{α})-strong porosity open sets and the (x^{α})-strong porosity topology - τ_{α}.

Obviously, all topologies T_{α} and τ_{α} are finer than the ρ-topology of the metric space (X,ρ). Put

$$T_{\alpha}^{*} = \{G \setminus P;\ G \text{ is } T_{\alpha}\text{-open and } P \text{ is a } T_{\alpha}\text{-first category set}\},$$

$$\tau_{\alpha}^{*} = \{G \setminus P;\ G \text{ is } \tau_{\alpha}\text{-open and } P \text{ is a } \tau_{\alpha}\text{-first category set}\}.$$

The systems T_{α}^{*} and τ_{α}^{*} form topologies (see [M]). They are called the $(x^{\alpha})^{*}$-porosity topology and the $(x^{\alpha})^{*}$-strong porosity topology.

The following propositions are analogous to Propositions 3-5 and Theorem 2 from Zajíček's paper [Z]. Their proofs are identical with those in [Z]. As usual, we assume that $\alpha \in (0,1]$, $G \subset X$ and $z \in X$.

PROPOSITION 2. If $z \in G$, then the following conditions are equivalent:

(i) G is a T_{α}-neighbourhood of z,

(ii) $\operatorname{int} G \cup \{z\}$ is a T_{α}-neighbourhood of z,

(iii) $X \setminus G$ is (x^{α})-superporous at z.

PROPOSITION 3. G is (x^{α})-porosity open if and only if there are an open set H and some $Z \subset \operatorname{Fr} H$, such that $G = H \cup Z$ and $X \setminus H$ is (x^{α})-superporous at each point of Z.

PROPOSITION 4. If G is (x^α)- porosity open, then $A \setminus \operatorname{int} A$ is (x^α)-superporous.

PROPOSITION 5. If (X,ρ) is a Baire space, then T_α^* is a category density topology on X, and

$$T_\alpha^* = \{G \setminus P;\ G \text{ is } T_\alpha\text{-open and } P \text{ is a } \rho\text{-first category set}\}.$$

REMARK 1. Evidently, analogues of Propositions 2-5 for the topologies $\tau_\alpha, \tau_\alpha^*$ and (x^α)-strongly superporous sets are also true.

It is evident that $T_\alpha \subsetneq T_\alpha^*$ and $\tau_\alpha \subsetneq \tau_\alpha^*$ for all $\alpha \in (0,1]$. In [F, Theorem 2 and 3] it was proved that if $X = R$, then no topology from the collection $\bigcup_{\alpha\in(0,1]} \{T_\alpha, \tau_\alpha\}$ $\left(\bigcup_{\alpha\in(0,1]} \{T_\alpha^*, \tau_\alpha^*\}\right)$ is included in any other topology from this collection. Now, we shall prove that all topologies T_α and τ_α are completely regular. We start with some properties of (x^α)-superporosity and (x^α)-strong superporosity.

PROPOSITION 6. Let $\alpha \in (0,1]$. If A is (x^α)-superporous at a point x_o, then there is an open set G (x^α)-superporous at x_o, including $A \setminus \{x_o\}$.

P r o o f. We may assume that x_o is not an isolated point of X, and that $A \subset B(x_o, (\frac{1}{2})^\alpha)$. Put

$$G = \bigcup_{x\in A\setminus\{x_o\}} B(x, \rho(x,x_o)^{(\alpha+1)/\alpha}).$$

Let F be (x^α)-porous at x_o. We must show that $G \cup F$ is (x^α)-porous at x_o. From the assumption it follows that $A \cup F$ is (x^α)-porous at x_o. Hence there are a positive number c and sequences of balls $B(x_n,r_n)$, $B(x_o,R_n)$, such that R_n tends decreasingly to 0 and

$$(1) \qquad B(x_n,r_n) \subset B(x_o,R_n) \setminus A \setminus F \quad \text{and} \quad \frac{r_n^\alpha}{R_n} > c$$

for every n. Since

$$\frac{(2R_n)^{(\alpha+1)/\alpha}}{r_n} = [\frac{2R_n}{r_n} (2R_n)^\alpha]^{1/\alpha} \xrightarrow[n\to\infty]{} 0,$$

there is a positive integer n_o such that $r_n > (2R_n)^{(\alpha+1)/\alpha}$ for $n \geq n_o$. Put

$$s_n = r_n - (2R_n)^{(\alpha+1)/\alpha}$$

for $n \geq n_o$. We shall show that

$$(2) \qquad B(x_n,s_n) \subset B(x_o,R_n) \setminus G \setminus F.$$

Suppose the contrary, i.e. there is $n \geq n_o$ such that $B(x_n,s_n) \cap G \neq \emptyset$. Let $y \in B(x_n,s_n) \cap G$. Then

$$\rho(x_n,y) < s_n, \qquad \rho(x_o,y) < R_n,$$

and there exists $x \in A \setminus \{x_o\}$ such that

$$\rho(x,y) < \rho(x,x_o)^{(\alpha+1)/\alpha}, \qquad \rho(x_o,x) < (\tfrac{1}{2})^\alpha.$$

Hence

$$R_n > \rho(x_o,y) \geq \rho(x_o,x) - \rho(x,y) > \rho(x,y)^{\alpha/(\alpha+1)} - \rho(x,y)$$

$$= \rho(x,y)(\frac{1}{\rho(x,y)^{1/(1+\alpha)}} - 1) > \rho(x,y)(\frac{1}{\rho(x,x_o)^{1/\alpha}} - 1)$$

$$\geq \rho(x,y)$$

and, consequently,

$$\rho(x_n,x) \le \rho(x_n,y) + \rho(x,y) < s_n + \rho(x,x_o)^{(\alpha+1)/\alpha}$$

$$\le s_n + (\rho(x,y) + \rho(y,x_o))^{(\alpha+1)/\alpha}$$

$$< s_n + (2R_n)^{(\alpha+1)/\alpha} = r_n.$$

This inequality contradicts condition (1) and thus proves condition (2).

Evidently, since $\frac{s_n}{r_n} \xrightarrow[n\to\infty]{} 1$, thus, for sufficiently large n, we have

(3) $$\frac{s_n^\alpha}{R_n} > c.$$

From (2) and (3) we conclude that $G \cup F$ is (x^α)-porous at x_o.

PROPOSITION 7. Let $\alpha \in (0,1]$. If A is (x^α)-strongly superporous at the point x_o, then there is an open set G (x^α)-strongly superporous at x_o, including $A \setminus \{x_o\}$.

P r o o f. To prove this proposition, it is sufficient to repeat the proof of Proposition 6, changing conditions (1) and (3) only. If $\alpha \in (0,1)$, then we replace these conditions by

(1´) $$B(x_n,r_n) \subset B(x_o,R_n) \setminus A \setminus F \quad \text{and} \quad \frac{r_n^\alpha}{R_n} > n,$$

(3´) $$\frac{s_n^\alpha}{R_n} > \frac{1}{2}n.$$

If $\alpha = 1$, then we put

(1″) $B(x_n, r_n) \subset B(x_o, R_n) \setminus A \setminus F$ and $\frac{r_n}{R_n} > \frac{1}{2} - \frac{1}{n}$,

(3″) $\frac{s_n}{R_n} = \frac{r_n}{R_n} - 4R_n > \frac{1}{2} - \frac{1}{n} - 4R_n \xrightarrow[n\to\infty]{} \frac{1}{2}$.

By a slight modification of the proof of Propositions 6 and 7 we get

PROPOSITION 8. Let $\alpha \in (0,1]$. If A is (x^α)-porous at x_o ((x^α)-strongly porous at x_o), then there is an open set G (x^α)-porous at x_o ((x^α)-strongly porous at x_o), including $A \setminus \{x_o\}$.

THEOREM 1. The topologies T_α and τ_α are completely regular for each $\alpha \in (0,1]$.

P r o o f. We prove the theorem for T_α (for τ_α the proof is similar). Evidently, T_α is a Hausdorff space (because it is finer than the ρ-topology). Let H be a T_α-closed set and $x_o \notin H$. This means that H is (x^α)-superporous at each point of $X \setminus H$. Thus H is (x^α)-superporous at x_o and, obviously, $\bar{H}$ is also (x^α)-superporous at x_o ($\bar{H}$ denotes the closure of H in the ρ-topology). By Proposition 6, it follows that there is an open set G superporous at x_o, including $\bar{H} \setminus \{x_o\}$. Put

$$f(x) = \begin{cases} 1; & x = x_o, \\ \dfrac{\operatorname{dist}(x,\bar{H})}{\operatorname{dist}(x,\bar{H}) + \operatorname{dist}(x, X \setminus G)}; & x \neq x_o. \end{cases}$$

It is easy to see that f is ρ-continuous at each point $x \neq x_o$. By Proposition 2, $(X \setminus G) \cup \{x_o\}$ is a T_α-neighbourhood of x_o.

Since $f(x) = 1$ for $x \in X \setminus G$, we conclude that f is T_α-continuous at x_o.

If (X,ρ) is a Baire space, then from Proposition 3 and Remark 1 it follows that T_α and τ_α are Baire spaces for all $\alpha \in (0,1]$. Thus Theorem D from [Z] implies that, under the above assumptions, a real function f is T_α-continuous (τ_α-continuous) if and only if it is T_α^*-continuous (τ_α^*-continuous). Therefore from Theorem 1 we get

THEOREM 2. Let (X,ρ) be a Baire space and $\alpha \in (0,1]$. Then T_α (τ_α) is the coarsest topology for which all T_α^*-continuous (τ_α^*-continuous) real functions are continuous.

References

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Received November 7, 1989