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Asymmetry of all Countable orders of a real function

Throughout this paper R will always denote the set of all real numbers: $\alpha, \beta, \gamma, \delta$ will stand for countable ordinal numbers greater or equal to one (If we use zero we shall state this fact explicitely.) and ω_1, as usual, will be the first uncountable ordinal.

The notion of a local system was introduced in [2], p. 3. Among numerous examples of local systems presented there one can find three which are the simplest and simultaneously the most interesting for us: S_∞ (page 4), S_∞^+ and S_∞^-. (Page 18 has indications how to define them.) We shall define three transfinite sequences of local systems: $\{\mathsf{S}_\infty^\alpha\}_{1 \leqq \alpha < \omega_1}$, $\{\mathsf{S}_\infty^{\alpha+}\}_{1 \leqq \alpha < \omega_1}$ and $\{\mathsf{S}_\infty^{\alpha-}\}_{1 \leqq \alpha < \omega_1}$ such that $\mathsf{S}_\infty^1 = \mathsf{S}_\infty$, $\mathsf{S}_\infty^{1+} = \mathsf{S}_\infty^+$ and $\mathsf{S}_\infty^{1-} = \mathsf{S}_\infty^-$. Recall that if $1 \leqq \alpha < \omega_1$ and $A \subset \mathsf{R}$, then the derivative of A of order α (A^α) is defined in the following way (see [1], pp. 261-262): if $\alpha = 1$, then $A^\alpha = A^1 = \{x : x \text{ is a point of accumulation of } A\}$; if $\alpha > 1$ and $\alpha = \beta + 1$ for some β, then $A^\alpha = (A^\beta)^1$; if α is a limit ordinal, then $A^\alpha = \bigcap_{1 \leqq \beta < \alpha} A^\beta$. It is known that the derivative of an arbitrary set $A \subset \mathsf{R}$ and of an arbitrary order α, $1 \leqq \alpha < \omega_1$ is a closed set and that: 1) for arbitrary $A \subset \mathsf{R}$ and $1 \leqq \alpha_1 < \alpha_2 < \omega_1$ we have $A^{\alpha_1} \supset A^{\alpha_2}$; 2) for every $A \subset \mathsf{R}$ there exists $\alpha < \omega_1$ such that $A^\alpha = A^{\alpha+1}$. (Hence $A^\alpha = A^\beta$ for each β such that $\alpha \leqq \beta < \omega_1$.)

Now let $\mathsf{S}_\infty^\alpha = \{\mathcal{S}_\infty^\alpha(x) : x \in \mathsf{R}\}$, where $\mathcal{S}_\infty^\alpha(x)$ is the family of all sets $S \subset \mathsf{R}$ such that $x \in S$ and $x \in S^\alpha$. Similarly, let $\mathsf{S}_\infty^{\alpha+} = \{\mathcal{S}_\infty^{\alpha+}(x) : x \in \mathsf{R}\}$ ($\mathsf{S}_\infty^{\alpha-} = \{\mathcal{S}_\infty^{\alpha-}(x) : x \in \mathsf{R}\}$), where $\mathcal{S}_\infty^{\alpha+}(x)$ ($\mathcal{S}_\infty^{\alpha-}(x)$) is the family of all sets $S \subset \mathsf{R}$ such that $x \in S$ and $x \in (S \cap [x, +\infty))^\alpha$ (resp. $(S \cap (-\infty, x])^\alpha$).

We shall consider only bounded real functions of a real variable. Let $f : \mathsf{R} \to \mathsf{R}$ be such a function. We shall say that $y \in \mathsf{R}$ is a limit number of order α, $1 \leqq \alpha < \omega_1$ of f at a point x if and only if y is a $(\mathsf{S}_\infty^\alpha)$-limit number of f at x, i.e. if and only if for every $\varepsilon > 0$ we have $\{x\} \cup f^{-1}((y-\varepsilon, y+\varepsilon)) \in \mathcal{S}_\infty^\alpha(x)$. Similarly we define right-hand (left-hand) limit numbers of order α of f at x using local system $\mathsf{S}_\infty^{\alpha+}$ ($\mathsf{S}_\infty^{\alpha-}$, respectively).

The set of all limit numbers (right-hand, left-hand limit numbers, respectively) of order α of f at x will be denoted by $L^\alpha(f,x)$ ($L^{\alpha+}(f,x)$, $L^{\alpha-}(f,x)$, respectively). We shall say that $x \in \mathsf{R}$ is a point of asymmetry of a function f of order α if and only if $L^{\alpha+}(f,x) \neq L^{\alpha-}(f,x)$. The set of all points of asymmetry of f of order α will be denoted by $As^\alpha(f)$.

The aim of this paper is to characterize the family $\{As^\alpha(f)\}_{1\leqq\alpha<\omega_1}$ for the class of bounded functions, f.

THEOREM 1. For every bounded function $f : \mathsf{R} \to \mathsf{R}$ there exists α, $1 \leqq \alpha < \omega_1$ such that $As^\alpha(f) = As^{\alpha+1}(f)$.

Proof. It suffices to show that there exists α, $1 \leqq \alpha < \omega_1$ such that for every $x \in \mathsf{R}$ we have $L^{\alpha+}(f,x) = L^{(\alpha+1)+}(f,x)$ and $L^{\alpha-}(f,x) = L^{(\alpha+1)-}(f,x)$.

Observe first that $y \in L^{\alpha+}(f,x)$ ($y \in L^{\alpha-}(f,x)$, respectively) if and only if for every open interval I with rational endpoints such that $y \in I$ we have $x \in (f^{-1}(I) \cap [x,+\infty))^\alpha$ ($x \in (f^{-1}(I) \cap (-\infty,x])^\alpha$, respectively). Let $\{I_1, I_2, \ldots, I_n, \ldots\}$ be a sequence of all intervals with rational endpoints. From property 2) of derivatives it follows that for each natural n there exists α_n, $1 \leqq \alpha_n < \omega_1$ such that $(f^{-1}(I_n))^{\alpha_n} = (f^{-1}(I_n))^{\alpha_n+1}$. Fix $n \in N$ and consider a family of sets $\{f^{-1}(I_n) \cap [x,+\infty) : x \in \mathsf{R}\}$. We shall show that for each $x \in \mathsf{R}$ $(f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n+1} = (f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n+2}$. Indeed, observe that $(f^{-1}(I_n))^{\alpha_n}$ is a perfect (possibly empty) set, because it is equal to its derivative. Observe also that if $z > x$, then $z \in (f^{-1}(I_n))^{\alpha_n}$ if and only if $z \in (f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n}$. Hence either $(f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n}$ is a perfect set and then $(f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n} = (f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n+1}$, or $(f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n}$ is a union of a perfect set and the singleton $\{x\}$ (which is isolated in $(f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n}$) and then $(f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n+1}$ is a perfect set. So, in both cases, the required equality holds for each $x \in \mathsf{R}$.

Obviously, the same reasoning gives $(f^{-1}(I_n) \cap (-\infty,x])^{\alpha_n+1} = (f^{-1}(I_n) \cap [x,+\infty))^{\alpha_n+2}$ for each $x \in \mathsf{R}$.

Let $\alpha < \omega_1$ be an ordinal number such that $\alpha_n + 1 < \alpha$ for each natural n. So we have $(f^{-1}(I_n) \cap [x,+\infty))^\alpha = f^{-1}(I_n) \cap [x,+\infty))^{\alpha+1}$ and $(f^{-1}(I_n) \cap (-\infty,x])^\alpha = (f^{-1}(I_n) \cap (-\infty,x])^{\alpha+1}$ for each $n \in N$ and each $x \in \mathsf{R}$. Hence $L^{\alpha+}(f,x) = L^{(\alpha+1)+}(f,x)$ and $L^{\alpha-}(f,x) = L^{(\alpha+1)-}(f,x)$ for each $x \in \mathsf{R}$, which completes the proof.

THEOREM 2. For each bounded function $f : \mathsf{R} \to \mathsf{R}$ and for every α, $1 \leqq \alpha < \omega_1$ the set $As^\alpha(f)$ is at most countable.

Proof. By virtue of Lemma 25.4 ([2], p. 59) the set $As^\alpha(f)$ is a subset of a set of the form

$$\bigcup_{n=1}^{\infty} (\mathsf{S}_\infty^{\alpha+} \Delta \mathsf{S}_\infty^{\alpha-}) - \mathrm{der}\,[A_n]$$

for some sequence of sets $\{A_n\}_{n \in N}$. Here $(\mathsf{S}) - \mathrm{der}\,[A] = \{x : \{x\} \cup A \in \mathcal{S}(x)\}$ for a local system S and $A \subset \mathsf{R}$ (see [2], p. 50) and $(\mathsf{S}_1 \Delta \mathsf{S}_2) - \mathrm{der}\,[A] = ((\mathsf{S}_1) - \mathrm{der}\,[A]) \Delta ((\mathsf{S}_2) - \mathrm{der}\,[A])$ for two local systems $\mathsf{S}_1, \mathsf{S}_2$ and $A \subset \mathsf{R}$ (see [2], p. 57). So it is sufficient to prove that for an arbitrary set $A \subset \mathsf{R}$ and for each α, $1 \leqq \alpha < \omega_1$ the set $((\mathsf{S}^{\alpha+}) - \mathrm{der}\,[A]) \Delta ((\mathsf{S}_\infty^{\alpha-}) - \mathrm{der}\,[A])$ is at most countable. We shall prove that $E_\alpha = ((\mathsf{S}_\infty^{\alpha+}) - \mathrm{der}\,[A]) - ((\mathsf{S}_\infty^{\alpha-}) - \mathrm{der}\,[A])$ is at most countable. (The proof for the second difference is similar.)

Observe first that $x \in (\mathsf{S}_\infty^{\alpha+}) - \mathrm{der}\,[A]$ ($x \in (\mathsf{S}_\infty^{\alpha-}) - \mathrm{der}\,[A]$) if and only if $x \in (A \cap [x, +\infty))^\alpha$ ($x \in (A \cap (-\infty, x])^\alpha$, respectively). Hence $E_\alpha = \{x \in \mathsf{R} : x \in (A \cap [x, +\infty))^\alpha - (A \cap (-\infty, x])^\alpha\}$. Suppose that E_γ is at most countable for all $\gamma < \alpha$. (This fact for $\gamma = 1$ is well-known.) If $\alpha = \beta + 1$ for some β, $1 \leqq \beta < \omega_1$, then $x \in E_\alpha$ if and only if $x \in ((A \cap [x, +\infty))^\beta)^1 - ((A \cap (-\infty, x])^\beta)^1$. Then for $x \in E_\alpha$ there exists $\delta_x > 0$ such that $(x - \delta_x, x) \cap (A \cap (-\infty, x])^\beta = \emptyset$ while for every $\delta > 0$ $(x, x + \delta) \cap (A \cap [x, +\infty))^\beta \neq \emptyset$. So $\{(x - \delta_x, x) : x \in E_\alpha\}$ is a disjoint family. Hence E_α is at most countable. If α is a limit number, then it is nearly obvious that $E_\alpha \subset \bigcup_{\gamma < \alpha} E_\gamma$, so E_α is also at most countable. This completes the proof.

THEOREM 3. If $x \in As^\alpha(f)$ and α is a limit ordinal, then there exists $\beta_0 < \alpha$ such that β_0 has a predecessor and $x \in \bigcap_{\beta_0 \leq \gamma \leq \alpha} As^\gamma(f)$. If $\beta_0 > 1$ and $\beta_0 = \delta + 1$, then $x \notin As^\delta(f)$.

Proof. Suppose that $L^{\alpha+}(f, x) - L^{\alpha-}(f, x) \neq \emptyset$. (The proof in the remaining case is similar.) Then there exists $y \in L^{\alpha+}(f, x) - L^{\alpha-}(f, x)$. Hence for every $\varepsilon > 0$ $x \in (f^{-1}((y - \varepsilon, y + \varepsilon)) \cap [x, +\infty))^\alpha$ and there exists $\varepsilon_0 > 0$ such that $x \notin (f^{-1}((y - \varepsilon_0, y + \varepsilon_0)) \cap (-\infty, x])^\alpha$. Then, according to the definition of the derivative of order α, there exists $\beta < \alpha$ such that $x \notin (f^{-1}((y - \varepsilon_0, y + \varepsilon_0)) \cap (-\infty, x])^\beta$. Hence, by virtue of property 1) of derivatives, $x \notin (f^{-1}((y - \varepsilon_0, y + \varepsilon_0)) \cap (-\infty, x])^\gamma$ for every γ, $\beta \leqq \gamma \leqq \alpha$. Likewise (again according to the definition of the derivative of order α) $x \in (f^{-1}((y - \varepsilon, y + \varepsilon)) \cap [x, +\infty))^\gamma$ for every $\varepsilon > 0$ and for $\beta \leqq \gamma \leqq \alpha$. Hence $x \in \bigcap_{\beta \leq \gamma \leq \alpha} As^\gamma(f)$. Let β_0 be the smallest ordinal number for which the last relation holds. If β_0 were a limit ordinal, the reasoning can be repeated resulting in $\beta < \beta_0$ for which again the relation holds.

So β_0 has a predecessor. The rest follows immediately.

Now we shall show that the conditions described in Theorems 1, 2 and 3 give the characterization of the family $\{As^\alpha(f)\}_{1\leqq\alpha<\omega_1}$ for bounded functions. We shall only slightly modify the formulation of the last condition.

THEOREM 4. If $\{E_\alpha\}_{1\leqq\alpha<\omega_1}$ is a family of sets of real numbers satisfying the following conditions:

1. there exists α_0, $1 \leqq \alpha_0 < \omega_1$ such that $E_\alpha = E_{\alpha_0}$ for each α, $\alpha_0 \leqq \alpha < \omega_1$;
2. for each α, $1 \leqq \alpha < \omega_1$ the set E_α is at most countable;
3. if $\alpha \leqq \alpha_0$ is a limit ordinal and $x \in E_\alpha$, then there exists $\gamma_\alpha < \alpha$ such that $x \notin E_{\gamma_\alpha}$ (if $\gamma_\alpha = 0$, put $E_0 = \emptyset$) and $x \in \bigcap_{\gamma_\alpha<\gamma\leqq\alpha} E_\gamma$,

then there exists a bounded function $f : \mathsf{R} \to \mathsf{R}$ such that $E_\alpha = As^\alpha(f)$ for each α, $1 \leqq \alpha < \omega_1$.

Proof. Before constructing the function f, we begin with some special partitions of sets of ordinal numbers associated with points from $\bigcup_{1\leqq\alpha<\omega_1} E_\alpha$. Let $x \in \bigcup_{1\leqq\alpha<\omega_1} E_a(= \bigcup_{1\leqq\alpha\leqq\alpha_0} E_\alpha)$. Put $H(x) = \{\alpha \leqq \alpha_0 : x \in E_\alpha\}$. Let $G(x) = \{\alpha \in H(x) : \alpha \text{ is a limit ordinal}\}$. Both $H(x)$ and $G(x)$ are at most countable. Let $G(x) = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$. By virture of condition 3 for each $n \in N$ there exists $\gamma\alpha_n < \alpha_n$. We shall construct a sequence of left-open intervals of ordinal numbers. Put $P_1(x) = \{\gamma : \gamma_{\alpha_1} < \gamma \leqq \alpha_1\}$. Obviously $P_1(x) \subset H(x)$. Suppose that we have already found disjoint intervals $P_1(x), P_2(x), \ldots, P_{n-1}(x)$. If $\alpha_n \in \bigcup_{i=1}^{n-1} P_i(x)$, then we put $P_n(x) = \emptyset$. If not, then let $\alpha_{i_1}, \ldots, \alpha_{i_k}$ be all ordinals among $\alpha_n, \ldots, \alpha_{n-1}$ which are smaller than α_n (if there are some) and put

$$P_n(x) = \{\gamma : \max(\gamma_{\alpha_n}, \alpha_{i_1}, \ldots, \alpha_{i_k}) < \gamma \leqq \alpha_n\}.$$

We have $P_n(x) \subset H(x)$ and $P_n(x) \cap P_i(x) = \emptyset$ for $i < n$. By induction we have defined a disjoint sequence $\{P_n(x)\}_{n\in N}$ of left-open intervals included in $H(x)$. Consider the set $H(x) - \bigcup_n P_n(x)$. This set is also at most countable, say $H(x) - \bigcup_n P_n(x) = \{\beta_1, \beta_2, \ldots, \beta_n, \ldots\}$. To unify the rest of the construction we shall artificially represent each singleton $\{\beta_n\}$ as a left-open interval. Observe first that each β_n has a predecessor, $\beta_n = \gamma_n + 1$. So $Q_n(x) = \{\beta_n\} = \{\gamma : \gamma_n < \gamma \leqq \beta_n\}$. Finally let $R_{2n-1}(x) = P_n(x), R_{2n}(x) = Q_n(x)$ for natural n. (If the family of nonempty P_n's or Q_n's is finite, we make the obvious change.) With each $x \in \bigcup_{1\leqq\alpha\leqq\alpha_0} E_\alpha$ we have associated a disjoint sequence of left-open nonempty intervals $\{R_n(x)\}_{n\in N}$ such that $H(x) = \bigcup_n R_n(x)$.

By virtue of condition 2 the set $\bigcup_{1\leqq\alpha\leqq\alpha_0} E_\alpha$ is at most countable. So the set of all ordered pairs $t = (x, R)$, where $x \in \bigcup_{1\leqq\alpha\leqq\alpha_0} E_\alpha$ and R is the interval associated with x is at most countable. Let $\{t_1, t_2, \ldots, t_n, \ldots\}$ be a sequence consisting of all ordered pairs described above.

We shall need some lemmas:

LEMMA 1. Let $x_0 \in \mathsf{R}$, let β_0, β_1 be countable ordinals such that $0 \leqq \beta_0 < \beta_1 < \omega_1$ and let $X \subset \mathsf{R}$ be an arbitrary countable set. There exists a countable set $F \subset \mathsf{R} - X$ such that for $h = \chi_F$ (the characteristic function of F) we have $As^\beta(h) = \{x_0\}$ for $\beta_0 < \beta \leqq \beta_1$ and $As^\beta(h) = \emptyset$ for $1 \leqq \beta \leqq \beta_0$ and for $\beta > \beta_1$.

Proof. First we construct a set $F^+ \subset [x_0, +\infty)$ fulfilling the following conditions: F^+ is a closed countable set, $x_0 \in F^+$, $(F^+ - \{x_0\}) \cap X = \emptyset$ and $As^\beta(\chi_{F^+} = \{x_0\}$ for $1 \leqq \beta \leqq \beta_1$, $As^\beta(\chi_{F^+}) = \emptyset$ for $\beta_1 < \beta < \omega_1$. More precisely, we shall have for $1 \leqq \beta \leqq \beta_1$

(1) $L^\beta(\chi_{F^+}, x) = L^{\beta+}(\chi_{F^+}, x) = L^{\beta-}(\chi_{F^+}, x) = \{0\}$ for $x \notin F^+$,

(2) $L^{\beta-}(\chi_{F^+}, x_0) = \{0\}, \quad L^{\beta+}(\chi_{F^+}, x_0) = \{0, 1\}$,

(3) $L^{\beta-}(\chi_{F^+}, x) = L^{\beta+}(\chi_{F^+}, x) = \{0, 1\}$ for $x \in (F^+)^1 - \{x_0\}$,

(4) $L^{\beta-}(\chi_{F^+}, x) = L^{\beta+}(\chi_{F^+}, x) = \{0\}$ for x isolated in F^+,

and for $\beta > \beta_1$ the only limit number (right-hand, left-hand limit number) of order β will be zero. For $\beta_1 = 1$ it suffices to take $F^+ = \{x_0, x_1, x_2, \ldots\}$, where the sequence $\{x_n\}_{n \in N}$ is strictly decreasing, convergent to x_0 and none of its elements belong to X.

Suppose now that for every $x \in \mathsf{R}$ and an arbitrary ordinal γ, $1 \leqq \gamma < \beta_1$, we can build a set with all the required properties and sufficiently small, i.e. included in a prescribed right-hand neighborhood of X. (This condition is not difficult to obtain.)

If $\beta_1 = \gamma + 1$, then let $\{x_n\}_{n \in N}$ be a decreasing sequence converging to x_0 and disjoint from X. Let F_n^+ be a set constructed for x_n and γ such that $F_n^+ \subset [x_n, x_n + \varepsilon_n)$, where the sequence $\{\varepsilon_n\}_{n \in N}$ is sufficiently small, that is $(x_n - \varepsilon_n, x_n + \varepsilon_n) \cap (x_{n+1} - \varepsilon_{n+1}, x_{n+1} + \varepsilon_{n+1}) = \emptyset$ for arbitrary $n \in N$. Moreover, each F_n^+ should be disjoint from X and, additionally $F_n^+ \cap (2x_n - X) = \emptyset$. (Here $a - X$ means the set $\{a - x : x \in X\}$.) Obviously for countable X the set $a - X$ is also countable.

Put $F_n^- = 2x_n - F_n^+$. We have $F_n^- \cap X = \emptyset$. Finally let $F^+ = \bigcup_{n \in N}(F_n^+ \cup F_n^-) \cup \{x_0\}$. It is not difficult to verify that F^+ fulfills all requirements.

If β_1 is a limit ordinal, then first we construct an increasing sequence $\{\gamma_n\}_{n \in N}$ of ordinals converging to β_1; then a decreasing sequence $\{x_n\}_{n \in N}$ converging to

x_0 and disjoint from X and then for each n F_n^+ is constructed for x_n and γ_n. The rest of the construction is without changes.

Now, if $\beta_0 = 0$, we put $K^- = \emptyset$. If $\beta_0 > 0$, then similarly as before we construct a set K^+ for x_0 and β_0 included in $[x_0, +\infty)$, disjoint from $(2x_0 - X)$. Finally we put $F = F^+ \cup (2x_0 - K^+) - \{x_0\}$.

The above set F fulfills all the conditions.

LEMMA 2. Let $x_0 \in \mathsf{R}$, and let β_0 be a countable ordinal greater or equal to 0 and let $X \subset \mathsf{R}$ be an arbitrary countable set. There exists a countable set $F \subset \mathsf{R} - X$ such that for $h = \chi_F$ we have $As^\beta(h) = \{x_0\}$ for $\beta_0 < \beta < \omega_1$ and $As^\beta(h) = \emptyset$ for $1 \leqq \beta \leqq \beta_0$.

Proof. Put $\beta_1 = \beta_0 + 1$. Let F_0 be a set constructed for x_0, β_0 and β_1 (from Lemma 1). Put $F = F_0 \cup G$, where $G \subset (\mathsf{R} - X) \cap (x_0, +\infty)$ is an arbitrary countable set dense in $(x_0, +\infty)$. The set F fulfills all the requirements.

Now we proceed to the construction of f:

Consider the pair $t_1 = (x, R)$. If $\alpha_0 \notin R$ (α_0 from condition 1), and $R = \{\gamma : \beta_0 < \gamma \leqq \beta_1\}$, then, using Lemma 1, we construct the set F_1 for x, β_0, β_1 and $X = \bigcup_{1 \leqq \alpha \leqq \alpha_0} E_\alpha$ and put $f_1 = \chi_{F_1}$. If $\alpha_0 \in R$, then, using Lemma 2, we construct the set F_1 for x, β_0 and $X = \bigcup_{1 \leqq \alpha \leqq \alpha_0} E_\alpha$ and again put $f_1 = \chi_{F_1}$.

Suppose that for $t_1, \ldots, t_{n-1}$ we have already found $F_1, \ldots, F_{n-1}$ and hence $f_1, \ldots, f_{n-1}$. Consider $t_n = (x, R)$. If $R = \{\gamma : \beta_0 < \gamma \leqq \beta_1\}$ and $\beta_1 < \alpha_0$, then we use Lemma 1 for x, β_0, β_1 and $X = \bigcup_{i=1}^{n-1} F_i \cup \bigcup_{1 \leqq \alpha \leqq \alpha_0} E_\alpha$ to find F_n. If $\beta_1 = \alpha_0$, we use Lemma 2 for x, β_0 and $X = \bigcup_{i=1}^{n-1} F_i \cup \bigcup_{1 \leqq \alpha \leqq \alpha_0} E_\alpha$ to find F_n. Next we put $f_n = \chi_{F_n}$.

So, by induction, we have defined a sequence of countable and disjoint sets $\{F_n\}_{n \in N}$ and a seqence of functions $\{f_n\}_{n \in N}$. Finally put $f = \sum_n \frac{1}{n} f_n$. The function f fulfills all the requirements.

References

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