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A Global Implicit Function Theorem

1. In this paper, $f(x,y)$ is a function of two variables defined on an open subset U of R^2. Let $D_1f(D_2f)$ denote the partial derivative of f with respect to the 1-st place (2-nd place) variable. We let $D_1^+f(D_2^+f)$ denote the upper right Dini derivate of f with respect to the 1-st place (2-nd place) variable. Likewise $D_1^-f(D_2^-f)$ denotes the upper left Dini derivate of f with respect to the 1-st place (2-nd place) variable.

As in [C] we say that the function f on U is locally bounded at a point $(x,y) \in U$ if f is bounded in some neighborhood of (x,y). It follows that f is locally bounded at (x,y) if f is continuous at (x,y).

The standard result on implicit functions for functions of two variables [Ct] is:

Theorem 0. Let f be continuously differentiable on U and let D_2f never vanish on U. Then any point $(x_0,y_0) \in U$ lies in a segment $I = \{(x,y_0) : a < x < b\}$ for which there is a differentiable function g defined on I such that $g(x_0) = y_0$, and $f(x,g(x)) = f(x_0,y_0)$ for $(x,y_0) \in I$; moreover, $g' = -D_1f/D_2f$ for $(x,y_0) \in I$.

In the spirit of [C], we offer a global theorem in which boundedness replaces continuity of the derivatives,

Theorem 1. Let f be a continuous function on U and let $D_1^+f, D_1^-f, D_2^+f, D_2^-f$ be each $< \infty$. Let $D_2^+f > 0$, and let $D_1^+(D_2^+f)$ be locally bounded on U. Then almost every point $(x_0,y_0) \in U$ lies in a segment $I = \{(x,y_0) \in U : a < x < b\}$ for which there is a continuous function g defined on I such that $g(x_0) = y_0$ and $f(x,g(x)) = f(x_0,y_0)$ for $(x,y_0) \in I$; moreover, at almost every point of I (relative to I) the derivatives g', D_1f and D_2f exist and $g' = -D_1f/D_2f$.

We also provide a variation, easier to prove, that employs Baire category instead of Lebesgue measure. We say that a set is residual if its complement is a first category set.

Proposition 1. Let f be a continuous function on U and let D_1f and D_2f exist on U. Let D_2f never vanish on U. Then there is a residual subset Z of U such that every point $(x_0,y_0) \in Z$ lies in a segment $I = \{(x,y_0) : a < x < b\}$

for which there is a continuous function g defined on I such that $g(x_0) = y_0$, and $f(x, g(x)) = f(x_0, y_0)$ for $(x, y_0) \in I$; moreover, there is a residual subset W of I (relative to I) such that at every point of W the derivative g' exists and $g' = -D_1 f / D_2 f$.

Our arguments will not depend on differentiability of the function f.

2. We begin with:

Lemma 1. Let E be a subset of U of measure zero, and let $E_0 = \{(p, y) \in U$: the set $\{w : (w, y) \in E\}$ does not have measure zero$\}$. Then $m(E_0) = 0$.

Proof. We deduce from the Fubini theorem applied to the characteristic function χ_E that the linear measure of the set of all y such that $\{(p, y) : (p, y) \in U\} \subset E_0$ has measure 0. We apply Fubini's theorem again to χ_{E_0} to prove that $m(E_0) = \int \chi_{E_0} = 0$. □

Lemma 2 does for category what Lemma 1 does for measure.

Lemma 2. Let E be a first category subset of U, and let $E_0 = \{(p, y) \in U :$ the set $\{w : (w, y) \in E\}$ is a second category set$\}$. Then E_0 is a first category set.

The proof is similar to the proof of Lemma 1 so we leave it. It also can be deduced from [LW, Lemma 2.2]. □

Lemma 3. Let $h(x)$ be a continuous function on an interval $[a, b]$ such that $D^+h(x)$ is real for all x. Let h be differentiable on a set $E \subset [a, b]$ such that $m([a, b] \setminus E) = 0$. Then

$$\inf_{x \in E} h'(x) \leq (h(b) - h(a))/(b - a) \leq \sup_{x \in E} h'(x).$$

Proof. Suppose to the contrary, that M is a real number such that

$$\sup_{x \in E} h'(x) < M < (h(b) - h(a))/(b - a).$$

Let $k(x) = h(x) - Mx$ on $[a, b]$. Then

$$\sup_{x \in E} k'(x) < 0 < (k(b) - k(a))/(b - a).$$

Moreover $D^+k(x)$ is real for all x, and by [S, p. 271], k maps sets of measure 0 to sets of measure 0. So $m(k([a, b] \setminus E)) = 0$. Now $k(b) > k(a)$. Select q such that $k(b) > q > k(a)$ and $q \notin k([a,b] \setminus E)$. Let x_0 be the greatest number in the compact set $k^{-1}(q)$. Then $x_0 \in E$ and $k'(x_0) < 0$. It follows that $k^{-1}(q)$ contains a number between x_0 and b by the intermediate value theorem, and this is impossible.

We conclude that $(h(b) - h(a))/(b-a) \leq \sup_{x \in E} h'(x)$. The other inequality is proved analogously with the signs reversed. $\square$

We return now to the function f in Theorem 1.

Lemma 4. Let f satisfy all the hypotheses of Theorem 1, and let u and v be nonzero numbers. Let the closed rectangle T with vertices $(x_0, y_0), (x_0 + u, y_0), (x_0, y_0 + v), (x_0 + u, y_0 + v)$ lie within U. Let $|D_1^+(D_2^+ f)| \leq M$ on T. Then

$$|f(x_0 + u, y_0 + v) + f(x_0, y_0) - f(x_0 + u, v_0) - f(x_0, y_0 + v)| \leq |uv|M.$$

Proof. Put $h(y) = f(x_0 + u, y) - f(x_0, y)$ over the interval joining y_0 and $y_0 + v$. Then

$$f(x_0 + u, y_0 + v) + f(x_0 + y_0) - f(x_0, y_0 + v) - f(x_0 + u, y_0) = h(y_0 + v) - h(y_0).$$

Now $0 < D_2^+ f(x_0 + u, y) < \infty$ and $0 < D_2^+ f(x_0, y) < \infty$, so $f(x_0 + u, y)$ and $f(x_0, y)$ are increasing with y. Thus $|D^+ h| < \infty$. Let E be the set of all y where $D_2 f(x_0 + u, y)$ and $D_2 f(x_0, y)$ exist. By [HS, pp. 264, 265]

$$m([\inf(y_0, y_0 + v), \sup(y_0, y_0 + v)] \setminus E) = 0.$$

By Lemma 3,

$$|v| \inf_{y \in E} h'(y) \leq |h(y_0 + v) - h(y_0)| \leq |v| \sup_{y \in E} h'(y).$$

But for any $y \in E, h'(y) = D_2 f(x_0 + u, y) - D_2 f(x_0, y), |D_2^+ f(x_0 + u, y) - D_2^+ f(x_0, y)| \leq |u|M$ and $|h'(y)| \leq |u|M$ by Dini's theorem [S, p. 204]. It follows that $|h(y_0 + v) - h(y_0)| \leq |uv|M$, and the conclusion follows. $\square$

Proof of Theorem 1. Let $(x_0, y_0) \in U$ and let $d > 0$. Then $f(x_0, y)$ increases with y because $D_2^+ f > 0$. Let y_1 and y_2 be such that $f(x_0, y_2) - f(x_0, y_1) < d, y_1 < y_0 < y_2, y_2 - y_1 < d$, and U contains the segment joining (x_0, y_1) to (x_0, y_2). Then $f(x_0, y_1) < f(x_0, y_0) < f(x_0, y_2)$. Let x_1 and x_2 be numbers such that $x_1 < x_0 < x_2, x_2 - x_1 < d$, the closed rectangle T with vertices $(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_1, y_2)$ lies within U, the function f exceeds $f(x_0, y_0)$ on the segment joining (x_1, y_2) to (x_2, y_2), $f(x_0, y_0)$ exceeds f on the segment joining (x_1, y_1) to (x_2, y_1), and such that the maximum of f on the segment joining (x_1, y_2) to (x_2, y_2) exceeds the minimum of f on the segment joining (x_1, y_1) to (x_2, y_1) by less than d. Now $f(x, y)$ increases with y for fixed x because $D_2^+ f > 0$. So for fixed x between x_1 and x_2 there is a unique y such that $f(x, y) = f(x_0, y_0)$ and $(x, y) \in T$. Let $g(x)$ denote this y; thus $f(x, g(x)) = f(x_0, y_0)$. This defines $g(x)$ for $x_1 < x < x_2$ and moreover, $|g(x) - g(x_0)| < d$ for any such x. Of course, $g(x_0) = y_0$.

To show that g is continuous, let $\varepsilon > 0$ and $x_1 < x < x_2$. By the same argument in the preceding paragraph with ε in place of d and a rectangle lying within T in place of T, we find an open interval J containing x such that if $u \in J$, then $|g(x) - g(u)| < \varepsilon$.

Now suppose that $D_1 f(x_0, y_0)$ and $D_2 f(x_0, y_0)$ exist. Then for $x \neq x_0$,

$$\begin{aligned} 0 &= f(x, g(x)) - f(x_0, y_0) = f(x, g(x)) - f(x, y_0) + f(x, y_0) - f(x_0, y_0) \\ &= (f(x_0, g(x)) - f(x_0, y_0)) + (f(x, y_0) - f(x_0, y_0)) \\ &\quad + (f(x, g(x)) - f(x, y_0) + f(x_0, y_0) - f(x_0, g(x)). \end{aligned}$$

If $|D_1^+(D_2^+ f)| < M$ on the rectangle with vertices $(x_0, y_0), (x, y_0), (x_0, g(x))$, $(x, g(x))$, then by Lemma 4 we have

$$(1) \quad |f(x_0, g(x)) - f(x_0, y_0) + f(x, y_0) - f(x_0, y_0)| \leq |(x - x_0)(g(x) - y_0)|M.$$

But $\lim_{x \to x_0} g(x) = y_0$, so there is a funtion $p(x)$ such that $\lim_{x \to x_0} p(x) = 0$ and

$$f(x_0, g(x)) - f(x_0, y_0) = (D_2 f(x_0, y_0) + p(x))(g(x) - y_0).$$

We deduce from (1) that

$$\begin{aligned} &|(D_2 f(x_0, y_0) + p(x))(g(x) - y_0)(x - x_0)^{-1} + (f(x, y_0) - f(x_0, y_0))(x - x_0)^{-1}| \\ (2) \qquad &\leq |g(x) - y_0|M. \end{aligned}$$

From (2) and the hypothesis that $D_1^+(D_2^+ f)$ is locally bounded and $D_2 f(x_0, y_0) > 0$, we deduce that $g'(x_0)$ exists and

$$(3) \qquad D_2 f(x_0, y_0) g'(x_0) + D_1 f(x_0, y_0) = 0.$$

We deduce from the continuity of f that the set of points where $D_1 f$ exists is measurable. By [S, pp. 270-271] $D_1 f(x, y)$ exists almost everywhere in x for each y. We deduce from the Fubini theorem applied to the characteristic function of the set of points where $D_1 f$ exists, that $D_1 f$ exists almost everywhere on U.

Likewise $D_2 f$ exists almost everywhere on U. Let E be a set such that $m(E) = 0$ and $D_1 f$ and $D_2 f$ exist on $U \setminus E$. Let E_0 be the set as in Lemma 1. We let $(x_0, y_0) \in U \setminus (E_0 \cup E)$ to obtain the desired conclusion. □

By a similar argument it can be shown that under the hypotheses of Theorem 1, f is in fact differentiable at any point where $D_1 f$ and $D_2 f$ exist. We leave this argument.

Before tackling the proof of Proposition 1 we offer an example. By [HS, p. 296] there is a measurable subset S of the real line R such that S and $R \setminus S$ meet any interval in sets of positive measure. Put

$$f(x,y) = x^6 y^5 + x^4 y^3 + \int_0^x \chi_S(t)dt + \int_0^y (1 + \chi_S(t))dt$$

for $(x,y) \in R^2$. Then f is a nontrivial function on R^2 satisfying the hypothesis of Theorem 1. But $D_1^+ f$ and $D_2^+ f$ are continuous at no point.

Proof of Proposition 1. By [Sg], $D_1 f$ and $D_2 f$ are continuous at all points of a residual subset of U. The proof is completed by arguing as in [Ct] and using Lemma 2 as Lemma 1 was used in the proof of Theorem 1. We leave the rest. □

References

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