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Topologies generated by porosity and strong porosity

1. INTRODUCTION

W. Wilczyński ([**W1**]) defined the $\mathcal{I}$-density topology $\mathcal{T}_{\mathcal{I}}$ on the real line which is a category analogue of the ordinary density topology on **R**. W. Poreda, E. Wagner-Bojakowska and W. Wilczyński ([**PBW**]) proved that the topology $\mathcal{T}_{\mathcal{I}}$ is not regular (unlike the density topology), but still, $\mathcal{I}$-approximately continuous ($=\mathcal{T}_{\mathcal{I}}$-continuous) functions are in the first class of Baire and have the Darboux property (like approximately continuous functions). The problem of finding the coarsest topology $\mathcal{T} \subset \mathcal{T}_{\mathcal{I}}$ which makes all $\mathcal{T}_{\mathcal{I}}$-continuous functions continuous was solved independently by W. Poreda and E. Wagner-Bojakowska ([**PB**]) and by E. Łazarow ([**Ł**]). For a survey of results concerning the $\mathcal{I}$-density topology see [**W2**].

Wilczyński's definition of the $\mathcal{I}$-density topology uses the algebraic structure of **R**. L. Zajíček in [**Z2**] introduces new topologies on a metric space (P, ϱ) using the notion of (ordinary) porosity: the porosity topology p and the *-porosity topology $p^\star$ where $p^\star$ is obtained by throwing sets of first category away from p-open sets. He shows ([**Z3**]) that the *-porosity topology on **R** is identical with the $\mathcal{I}$-density topology. He also studies some properties of these topologies (see [**Z2**], [**Z3**]). If P is a Baire space, then the class of all p-continuous functions is equal to the class of all $p^\star$-continuous functions, and these functions are in the first class of Baire on P. The topology $p^\star$ is determined by a category lower density.

There are several variants of the notion of ordinary porosity: strong porosity, (g)-porosity, $\langle H\rangle$-porosity (definitions can be found e.g. in [**Z1**] or [**Z4**]). Replacing the notion of porosity in the definition of the topologies p and $p^\star$ by these variants, definitions of new topologies and the corresponding *-topologies can be obtained. For example, strong porosity leads to the strong porosity topology s and the *-strong porosity topology $s^\star$. L. Zajíček ([**Z2**]) remarks that all these *-topologies have similar properties, in particular, they are determined by a category lower density.

The purpose of the present paper is to prove some properties of the topologies s and $s^\star$. We restrict ourselves to the case of a (real) normed linear space Q. Moreover, we add some new results on the topology p. It turns out that on a non-trivial space Q neither s is finer than p (which is almost obvious) nor p is finer than s (which may seem to be surprising). We also show that both the porosity topology p on P and the strong porosity topology s on Q possess the (complete) Lusin-Menchoff property. As a consequence of this fact we obtain that these topologies are completely regular. It follows that p is the coarsest topology which makes all $p^\star$-continuous functions continuous, therefore on **R** the porosity topology p coincides with the topology $\mathcal{T}$ introduced by E. Łazarow. Using the idea of [**Z2**] we prove that $s^\star$-continuous functions on a Hilbert space Q are in the first class of Baire. In the final part of the paper we investigate membership of $p^\star$-continuous and $s^\star$-continuous functions on **R** in the classes of Zahorski.

2. Definitions, basic properties

Let (P, ϱ) be a metric space. The symbol $U(x, r)$ will stand for the open ball centered at x with the radius r. For $x \in P$, $M \subset P$ and $R > 0$ put

$$\gamma(M, x, R) = \sup(\{r > 0 : \text{there is } y \in P \text{ such that } U(y, r) \subset U(x, R) \setminus M\} \cup \{0\}).$$

The number

$$\pi(M, x) = \limsup_{R \to 0+} \frac{\gamma(M, x, R)}{R}$$

is called the *porosity of* M *at* x.

We say that a set $M \subset P$ is *porous at a point* $x \in P$ if $\pi(M, x) > 0$. We say that M is *strongly porous at* x if $\pi(M, x) \geq \frac{1}{2}$.

A set $E \subset P$ is said to be *superporous at a point* $x \in P$ provided that $E \cup F$ is porous at x whenever $F \subset P$ is porous at x. Similarly, E is said to be *strongly superporous at* x provided that $E \cup F$ is strongly porous at x whenever F is strongly porous at x.

It is obvious that the collection of all sets which are superporous at a point $x \in P$ is an ideal (unlike the collection of all sets which are porous at x). The same is true of the collection of all sets which are strongly superporous at x. It follows that the systems

$$p = \{G \subset P : P \setminus G \text{ is superporous at any point } x \in G\}$$

and

$$s = \{G \subset P : P \setminus G \text{ is strongly superporous at any point } x \in G\}$$

form topologies on P which are finer than the initial topology. The topology p is called the *porosity topology*, s is called the *strong porosity topology*.

The following descriptions of p-neighbourhoods and s-neighbourhoods of a point $x \in P$ are simple consequences of the above mentioned fact that all sets which are superporous (strongly superporous) at x form an ideal and of the obvious fact that a set M is superporous (strongly superporous) at x iff $\overline{M}^{\varrho}$ is superporous (strongly superporous) at x.

Proposition 1a ([**Z2**, Proposition 3]). *Let $V \subset P$, $x \in V$. Then the following conditions are equivalent.*

(i) *V is a p-neighbourhood of x.*
(ii) *$\operatorname{int}_\varrho V \cup \{x\}$ is a p-open p-neighbourhood of x.*
(iii) *$P \setminus V$ is superporous at x.*

Proposition 1b. *Let $V \subset P$, $x \in V$. Then the following conditions are equivalent.*

(i) *V is an s-neighbourhood of x.*
(ii) *$\operatorname{int}_\varrho V \cup \{x\}$ is an s-open s-neighbourhood of x.*
(iii) *$P \setminus V$ is strongly superporous at x.*

For the points of ordinary porosity of a set $M \subset P$ we have the following simple characterization.

LEMMA 1a ([**Z2**, Lemma 2]). *Let $M \subset P$, $x \in P$. If x is an isolated point of P, then M is porous at x iff $x \notin M$. If x is not an isolated point of P, then M is porous at x iff there exists $c > 0$ and sequences of balls $\{U(x, R_n)\}$, $\{U(y_n, r_n)\}$ such that*

$$\lim R_n = 0, \quad \frac{r_n}{R_n} > c, \quad x \notin U(y_n, r_n), \quad U(y_n, r_n) \subset U(x, R_n) \setminus M.$$

We shall investigate properties of the strong porosity topology on a normed linear space Q with a norm $\|\ldots\|$. It is obvious that a set $E \subset Q$ is porous (superporous, strongly porous, strongly superporous, respectively) at a point $x \in Q$ iff the set $E - x = \{y - x : y \in E\}$ is porous (superporous, strongly porous, strongly superporous, respectively) at the point $\mathbf{0} \in Q$. Consequently, $x \in \operatorname{int}_p E$ iff $\mathbf{0} \in \operatorname{int}_p(E - x)$, and $x \in \operatorname{int}_s E$ iff $\mathbf{0} \in \operatorname{int}_s(E - x)$.

As an analogue of Lemma 1a we state

LEMMA 1b. *A set $M \subset Q$ is strongly porous at $\mathbf{0}$ if and only if there exists a sequence $\{U(y_n, r_n)\}$ of balls such that*

$$\mathbf{0} \notin U(y_n, r_n), \quad U(y_n, r_n) \cap M = \emptyset, \quad \lim y_n = \mathbf{0}, \quad \lim \frac{r_n}{\|y_n\|} = 1.$$

L. Zajíček characterized the p-interior points of a set $V \subset P$.

PROPOSITION 2a ([**Z2**, Proposition 8]). *Let $V \subset P$, $x \in V$. Then V is a p-neighbourhood of x (equivalently, $P \setminus V$ is superporous at x) if and only if the following condition (C_p) is satisfied.*

For any $u > 0$ there exists $d > 0$ and $v > 0$ such that whenever $U(x, R)$, $U(y, r)$ are balls for which $U(y, r) \subset U(x, R) \setminus \{x\}$, $R < d$, $r/R > u$, then there exists a ball $U(z, a) \subset U(y, r) \cap V$ such that $a/r > v$.

In a normed linear space Q, Proposition 2a reads as follows. (The proof requires simple technique only and we shall omit it.)

PROPOSITION 2a$'$. *Let $V \subset Q$, $\mathbf{0} \in V$. Then V is a p-neighbourhood of $\mathbf{0}$ (equivalently, $Q \setminus V$ is superporous at $\mathbf{0}$) if and only if the following condition $(C_p{'})$ is satisfied.*

For any $c \in (0, 1)$ there exists $\delta > 0$ and $\varepsilon \in (0, 1)$ such that whenever $x \in Q$, $0 < \|x\| < \delta$, then there exists $y \in Q$ and $r > 0$ such that $U(y, r) \subset U(x, c\|x\|) \cap V$, $r \geq \varepsilon\|y\|$.

Our aim now is to prove a similar characterization of the s-interior points of a set $V \subset Q$.

PROPOSITION 2b. *Let $V \subset Q$, $\mathbf{0} \in V$. Then V is an s-neighbourhood of $\mathbf{0}$ if and only if the following condition (C_s) is satisfied.*

For any $\varepsilon \in (0, 1)$ there exists $\delta > 0$ and $c \in (0, 1)$ such that whenever $x \in Q$, $0 < \|x\| < \delta$, then there exists $y \in Q$ and $r > 0$ such that $U(y, r) \subset U\big(x, (1-c)\|x\|\big) \cap V$, $r \geq (1 - \varepsilon)\|y\|$.

PROOF: (a) Let V satisfy the condition (C_s), let F be strongly porous at $\mathbf{0}$. We shall prove that $(Q\setminus V)\cup F$ is strongly porous at $\mathbf{0}$. For $\varepsilon_n = \frac{1}{n+1}$ we find the corresponding δ_n, c_n from the condition (C_s). By Lemma 1b there exists a sequence of balls $\{ U_n : n = 1,2,\dots \}$ where $U_n = U(x_n, R_n)$ such that $\mathbf{0} \notin U_n$, $U_n \cap F = \emptyset$, $\lim x_n = \mathbf{0}$, $\lim R_n/\|x_n\| = 1$. Let $\{\sigma(n)\}$ be an increasing sequence of natural numbers such that $\|x_{\sigma(n)}\| < \delta_n$, $R_{\sigma(n)} \geq (1-c_n)\|x_{\sigma(n)}\|$. The condition (C_s) then guarantees that there exists a sequence of balls $U(y_n, r_n)$ such that

$$U(y_n, r_n) \subset U(x_{\sigma(n)}, (1-c_n)\|x_{\sigma(n)}\|) \cap V \subset U_{\sigma(n)} \cap V \subset U_{\sigma(n)} \setminus [(Q \setminus V) \cup F],$$

$r_n \geq (1-\varepsilon_n)\|y_n\|$. This sequence of balls satisfies the condition of Lemma 1b for the set $M = (Q \setminus V) \cup F$. We have shown that $Q \setminus V$ is strongly superporous at $\mathbf{0}$ and therefore by Proposition 1b, V is an s-neighbourhood of $\mathbf{0}$.

(b) Assume that $Q \setminus V$ is strongly superporous at $\mathbf{0}$ and (C_s) does not hold. Then there exists $\varepsilon \in (0,1)$ such that for any $\delta > 0$, $c > 0$ there is $x \in Q$, $0 < \|x\| < \delta$ having the property that whenever $U(y,r) \subset U(x, (1-c)\|x\|) \cap V$, then $r < (1-\varepsilon)\|y\|$. It follows that for the sequence $\{c_n\}$ where $c_n = \frac{1}{n+1}$ it is possible to construct by induction a sequence $\{U_n\}$ of pairwise disjoint balls where $U_n = U(x_n, (1-c_n)\|x_n\|)$, $\lim x_n = \mathbf{0}$ such that for any ball $U(y,r) \subset U_n \cap V$ we have $r < (1-\varepsilon)\|y\|$. Put

$$F = Q \setminus \bigcup_{n=1}^{\infty} U_n.$$

Since F is strongly porous at $\mathbf{0}$, so is $(Q \setminus V) \cup F$. By Lemma 1b there is a sequence of balls $\{ U(y_k, r_k) : k = 1,2,\dots \}$ such that $\lim r_k/\|y_k\| = 1$ and

$$U(y_k, r_k) \subset Q \setminus [(Q \setminus V) \cup F] = \bigcup_{n=1}^{\infty} U_n \cap V.$$

Since the balls U_n are pairwise disjoint, for any k there is n such that $U(y_k, r_k) \subset U_n \cap V$. But then we obtain that $r_k < (1-\varepsilon)\|y_k\|$ for any k, which is a contradiction. ∎

We close this section by giving two examples which show that there is no connection between the porosity topology p and the strong porosity s on an arbitrary normed linear space $Q \neq \{\mathbf{0}\}$. If a set M is strongly porous at a point x, then a fortiori M is porous at x (the opposite assertion being false). So one might conjecture at first glance that strong superporosity of M at x implies superporosity of M at x. However, this conjecture is not justified.

EXAMPLE 1. *Let $Q \neq \{\mathbf{0}\}$ be a normed linear space. Then the set*

$$A = \bigcup_{n=0}^{\infty} \{ x \in Q : \|x\| = \frac{1}{2^n} \}$$

is superporous at $\mathbf{0}$ but not strongly superporous at $\mathbf{0}$. Consequently, $Q \setminus A$ is p-open but not s-open.

PROOF: (a) It is obvious that $\pi(A, \mathbf{0}) = \frac{1}{4}$. Therefore A is not strongly porous at $\mathbf{0}$ and, a fortiori, not strongly superporous at $\mathbf{0}$.

(b) To show that A is superporous at $\mathbf{0}$ we prove that $Q \setminus A$ satisfies the condition $(C_p{}')$ of Proposition 2a$'$. Let $c \in (0,1)$ be given. Put $c^* = \min(c, \frac{1}{3})$, $\varepsilon = c^*/2$. From the inequality $(1-c^*)/(1+c^*) \geq \frac{1}{2}$ it follows that for any $x \in Q \setminus \{\mathbf{0}\}$ the interval

$$\big((1-c^*)\|x\|,\, (1+c^*)\|x\|\big)$$

can contain at most one number $1/2^n$. Consequently, putting $y_1 = (1-c^*/2)\,x$ and $y_2 = (1+c^*/2)\,x$, we have that either $U(y_1, \varepsilon\|x\|)$ or $U(y_2, \varepsilon\|x\|)$ is contained in $U(x, c\|x\|) \setminus A$. The value for $\delta > 0$ in the condition $(C_p{}')$ can be chosen arbitrarily. ∎

EXAMPLE 2. *Let $Q \neq \{\mathbf{0}\}$ be a normed linear space. Let $\{r_n\}$ be a decreasing sequence of positive real numbers such that $\lim r_n = 0$, $\lim r_{n+1}/r_n = 0$. Then the set*

$$B = \bigcup_{n=1}^{\infty} \{\, x \in Q : r_n \leq \|x\| \leq 2r_n \,\}$$

is strongly superporous at $\mathbf{0}$ but not superporous at $\mathbf{0}$. Consequently, $Q \setminus B$ is s-open but not p-open.

PROOF: (a) In view of the fact that $Q \setminus B$ is porous at $\mathbf{0}$, B cannot be superporous at $\mathbf{0}$.

(b) First we prove that the set

$$B^* = \bigcup_{n=1}^{\infty} [\, r_n,\, 2r_n \,] \subset \mathbf{R}$$

satisfies the following condition:

> For any $\vartheta \in (0,1)$ there exists $\delta > 0$ and $c \in (0,1)$ such that whenever $0 < z < \delta$, then there exists an interval $(a,b) \subset (cz, z) \setminus B^*$ such that $a = \vartheta\, b$.

Choose $\vartheta \in (0,1)$. There is an index n_0 such that $2r_n/r_{n-1} < \vartheta^2/2$ for any $n \geq n_0$. Put $\delta = (4/\vartheta^2)\, r_{n_0}$, $c = \vartheta^2/2$. Choose $z \in \mathbf{R}$ such that $0 < z < \delta$. Let n be the greatest index for which $2r_n > (\vartheta^2/2)\, z$ (obviously $n \geq n_0$). We distinguish two cases.

1. Assume that $(\vartheta^2/2)\, z < 2r_n \leq \vartheta z$. If the inequality $r_{n-1} < z$ were satisfied, then we would obtain that $2r_n/r_{n-1} > \big[(\vartheta^2/2)\, z\big]/z = \vartheta^2/2$, which is impossible. Hence

$$(\vartheta z, z) \subset (cz, z) \cap (2r_n, r_{n-1}) \subset (cz, z) \setminus B^*.$$

2. Assume that $2r_n > \vartheta z$. Then, since $2r_{n+1} \leq (\vartheta^2/2)\, z$, it follows that

$$\Big(\frac{\vartheta^2}{2}\, z,\, \frac{\vartheta}{2}\, z\Big) \subset (cz, z) \cap (2r_{n+1}, r_n) \subset (cz, z) \setminus B^*.$$

Now it is easy to finish the proof by showing that $Q \setminus B$ satisfies the condition (C_s) of Proposition 2b. Choose $\varepsilon \in (0,1)$. Put $\vartheta = \varepsilon/(2-\varepsilon)$ and find the corresponding $\delta > 0$, $c \in (0,1)$ by the above condition. Choose $x \in Q$ such that $0 < \|x\| < \delta$. Find an interval $(a,b) \subset (c\|x\|, \|x\|) \setminus B^*$ for which $a = \vartheta\, b$. Putting $y = [\,(a+b)/(2\|x\|)\,]\, x$, $r = (b-a)/2$ we obtain

$$U(y, r) \subset U\big(x,\, (1-c)\|x\|\big) \setminus B,$$
$$r = \frac{b-a}{a+b}\,\|y\| = \frac{1-\vartheta}{1+\vartheta}\,\|y\| = (1-\varepsilon)\|y\|,$$

as required. ∎

To conclude these observations, we state

PROPOSITION 3. *Let p, s be the porosity topology and the strong porosity topology on a normed linear space $Q \neq \{0\}$, respectively. Then neither s is finer than p nor p is finer than s.*

3. The Lusin-Menchoff property

Recall that a fine topology τ on a topological space (X, ϱ) has the *Lusin-Menchoff property (with respect to ϱ)* if the following condition is satisfied.

For any pair of sets $F, H \subset X$ such that F is ϱ-closed, H is τ-closed, $F \cap H = \emptyset$ there are sets $G, W \subset X$ such that G is ϱ-open, W is τ-open, $F \subset W$, $H \subset G$ and $G \cap W = \emptyset$.

We say that τ has the *complete Lusin-Menchoff property (w.r.t. ϱ)* if for any subspace $Y \subset X$, τ induced on Y has the Lusin-Menchoff property w.r.t. the topology ϱ induced on Y.

We summarize some well-known facts on the Lusin-Menchoff property (see [**LMZ**, Chapter 3.B]).

THEOREM A. *Let τ be a fine topology on a topological space (X, ϱ) having the Lusin-Menchoff property w.r.t. ϱ. Then for any τ-open set $E \subset X$ of type F_σ there exists a τ-continuous ϱ-upper-semicontinuous function f on X such that*

$$0 < f(x) \leq 1 \quad \text{for all } x \in E \qquad \text{and} \qquad f(x) = 0 \quad \text{for all } x \in X \setminus E.$$

THEOREM B. *Any fine topology τ on a T_1-space (X, ϱ) having the Lusin-Menchoff property w.r.t. ϱ is completely regular.*

THEOREM C. *A fine topology τ on (X, ϱ) has the complete Lusin-Menchoff property w.r.t. ϱ if and only if the following condition is satisfied.*

For any pair of sets $A, B \subset X$ such that $\overline{A}^{\varrho} \cap B = A \cap \overline{B}^{\tau} = \emptyset$ there are sets $G, W \subset X$ such that G is ϱ-open, W is τ-open, $A \subset W$, $B \subset G$ and $G \cap W = \emptyset$.

THEOREM 1a. *The porosity topology p on a metric space (P, ϱ) has the complete Lusin-Menchoff property (w.r.t. the initial topology).*

PROOF: Assume that $\emptyset \neq A, B \subset P$ are biseparated sets, i.e. $\overline{A}^{\varrho} \cap B = A \cap \overline{B}^{p} = \emptyset$. Putting

$$G = \{\, x \in P : \varrho(x, B) < \varrho^2(x, A) \,\}, \quad W = P \setminus \overline{G}^{p} = \operatorname{int}_p(P \setminus G),$$

we see that G is ϱ-open, W is p-open, $G \cap W = \emptyset$, $B \subset G$ and $A \subset P \setminus G$. It remains to show that $A \subset W = \operatorname{int}_p(P \setminus G)$. To this end, we shall verify that $P \setminus G$ satisfies the condition (C_p) of Proposition 2a at any point $x \in A$. Fix $x \in A$. Then $x \in P \setminus \overline{B}^{p} = \operatorname{int}_p(P \setminus B)$. By Proposition 2a, $P \setminus B$ satisfies the condition (C_p) at x. Choose $u > 0$ and find by (C_p) the corresponding $d > 0$, $v > 0$. Put $d^* = \min(d, uv/2)$, $v^* = v/2$. Let $U(y, r) \subset U(x, R)$ be balls for which $x \notin U(y, r)$, $R < d^*$ and $r/R > u$. Then there is a ball $U(z, a) \subset U(y, r) \setminus B$ with $a/r > v$. For an arbitrary point $\eta \in G \cap U(y, r)$ we have

$$\varrho(\eta, B) < \varrho^2(\eta, A) \leq \varrho^2(\eta, x) < R^2 = \frac{R}{r} r R < \frac{a}{uv} \cdot d^* \leq \frac{a}{uv} \cdot \frac{uv}{2} = \frac{a}{2},$$
$$\varrho(z, \eta) \geq \varrho(z, B) - \varrho(\eta, B) \geq a - \frac{a}{2} = \frac{a}{2},$$

consequently $U(z, a/2) \subset U(y,r) \setminus G$. Since $(a/2)/r > v/2 = v^*$, the validity of the condition (C_p) at x for $P \setminus G$ is verified, and the proof is complete. ∎

COROLLARY. *The porosity topology p on a metric space (P, ϱ) is completely regular.*

THEOREM 1b. *The strong porosity topology s on a normed linear space Q has the complete Lusin-Menchoff property (w.r.t. the initial topology).*

PROOF: Assume that A, B are nonempty subsets of Q such that $\overline{A} \cap B = A \cap \overline{B}^s = \emptyset$. Put

$$G = \{ x \in Q : \operatorname{dist}(x, B) < \operatorname{dist}^2(x, A) \}, \quad W = Q \setminus \overline{G}^s = \operatorname{int}_s(Q \setminus G).$$

Then obviously G is open, W is s-open, $G \cap W = \emptyset$, $B \subset G$ and $A \subset Q \setminus G$. By Theorem C it remains to prove that $A \subset W = \operatorname{int}_s(Q \setminus G)$. Let $x \in A$ be given. To simplify the notation, we shall assume that $x = 0$, the idea in the general case remaining clear. Now it suffices to show that $Q \setminus G$ satisfies the condition (C_s) of Proposition 2b.

Choose $\varepsilon \in (0,1)$. Since $0 \in Q \setminus \overline{B}^s = \operatorname{int}_s(Q \setminus B)$, the condition (C_s) holds for $Q \setminus B$. Find $\delta > 0$, $c \in (0,1)$ which correspond to $\varepsilon/2$ by (C_s). Put

$$\delta^* = \min\Big(\delta, \frac{\varepsilon c}{8}\Big). \tag{1}$$

Choose $\xi \in Q$ such that $0 < \|\xi\| < \delta^*$. Then there exists a ball $U(\eta, r)$ contained in $U(\xi, (1-c)\|\xi\|) \setminus B$ with $r \geq (1 - \varepsilon/2)\|\eta\|$. We have

$$\|\eta\| \geq \|\xi\| - \|\eta - \xi\| > \|\xi\| - (1-c)\|\xi\| = c\|\xi\|. \tag{2}$$

Let z be an arbitrary point in $U(\eta, r) \cap G$. Then

$$\|z\| \leq \|z - \xi\| + \|\xi\| < 2\|\xi\|,$$
$$\operatorname{dist}(z, B) < \operatorname{dist}^2(z, A) \leq \|z\|^2 < 4\|\xi\|^2,$$

hence

$$\|z - \eta\| \geq \operatorname{dist}(\eta, B) - \operatorname{dist}(z, B) > r - 4\|\xi\|^2. \tag{3}$$

It follows from (1) and (2) that

$$4\|\xi\|^2 \leq \frac{4}{c}\,\|\xi\| \cdot \|\eta\| < \frac{4}{c}\,\delta^*\|\eta\| \leq \frac{\varepsilon}{2}\,\|\eta\|. \tag{4}$$

Finally, combining (3), (4) we obtain

$$\|z - \eta\| > r - 4\|\xi\|^2 > (1 - \frac{\varepsilon}{2})\|\eta\| - \frac{\varepsilon}{2}\|\eta\| = (1 - \varepsilon)\|\eta\|,$$

which implies that

$$U(\eta, (1-\varepsilon)\|\eta\|) \subset U(\eta, r) \setminus G \subset U(\xi, (1-c)\|\xi\|) \setminus G.$$

By Proposition 2b, $0 \in \operatorname{int}_s(Q \setminus G) = W$. ∎

COROLLARY. *The strong porosity topology s on a normed linear space Q is completely regular.*

4. *-POROSITY TOPOLOGIES

Recall that a topological space X is called a *Baire space* if the Baire Category Theorem holds in X, i.e. if $X \setminus A$ is dense in X whenever $A \subset X$ is of first category in X (equivalently, if any nonempty open subset of X is of second category in X).

We say that two topologies τ, σ on X are *S-related* provided that for any set $A \subset X$, $\operatorname{int}_\tau A \neq \emptyset$ iff $\operatorname{int}_\sigma A \neq \emptyset$. For such topologies the notions of dense sets, nowhere dense sets, sets of first category coincide. Moreover, (X,τ) is a Baire space iff (X,σ) is a Baire space.

Let (X,τ) be a Baire space. It is not hard to prove (see e.g. [**LMZ**]) that the collection

$$\tau^\star = \{\, G \setminus N : G \text{ is } \tau\text{-open, } N \text{ is of first category in } \tau \,\}$$

forms a topology. Obviously, $\tau^\star$ is finer than τ. If, moreover, σ is a topology on X which is S-related to τ, then we have

$$\tau^\star = \{\, G \setminus N : G \text{ is } \tau\text{-open, } N \text{ is of first category in } \sigma \,\}.$$

In what follows, (P,ϱ) will be a Baire space.

It is clear from Propositions 1a, 1b that both the porosity topology p and the strong porosity topology s on P are S-related to the ϱ-topology. The corresponding topology $p^\star$ is called the **-porosity topology*, $s^\star$ is called the **-strong porosity topology*.

THEOREM D ([**Z2**]). *Let (X,τ) be a Baire space and let f be a real function on X. Then f is $\tau^\star$-continuous on X if and only if f is τ-continuous on X.*

COROLLARY. *Let f be a real function on a Baire metric space (P,ϱ). Then f is $p^\star$-continuous on P iff f is p-continuous on P, and f is $s^\star$-continuous on P iff f is s-continuous on P.*

A topology τ on a metric space (P,ϱ) is said to satisfy the *essential radius condition* if for each $x \in P$ and each τ-neighbourhood U of x there is an "essential radius" $r(x,U) > 0$ such that whenever

$$0 < \varrho(x,y) \le \min(r(x,U_x), r(y,U_y)),$$

then $U_x \cap U_y \neq \emptyset$.

THEOREM E ([**LMZ**, pp. 64 and 66]). *Let (P,ϱ) be a metric space, let τ be a topology on P which satisfies the essential radius condition. Then any τ-continuous function on P is in the first class of Baire.*

THEOREM 2a ([**Z2**, Theorem 3]). *If (P,ϱ) is a Baire space, then the *-porosity topology $p^\star$ on P satisfies the essential radius condition.*

COROLLARY. *Any $p^\star$-continuous function on a Baire metric space (P,ϱ) is in the first class of Baire.*

LEMMA 2. *Let Q be a Hilbert space. If $\alpha > 1$, $u,v \in Q$, $\|u\| > \|v\|$, then $\|\alpha u - v\| > \|u - v\|$.*

THEOREM 2b. *Let Q be a Hilbert space. Then the *-strong porosity topology $s^\star$ on Q satisfies the essential radius condition.*

PROOF: For $z \in Q$ and an $s^\star$-neighbourhood $W^\star$ of z we shall determine an essential radius $r(z, W^\star)$ in the following way. Choose an s-neighbourhood W of z such that $W \setminus W^\star$ is of first category. Then $W - z$ is an s-neighbourhood of 0. By Proposition 2b, the following condition is satisfied:

For any $\varepsilon \in (0,1)$ there is $\delta > 0$ and $c \in (0,1)$ such that whenever $\xi \in Q$, $0 < \|\xi - z\| < \delta$, then there exists a ball $U(\eta, r) \subset U(\xi, (1-c)\|\xi - z\|) \cap W$ for which $r \geq (1-\varepsilon)\|\eta - z\|$.

Find to $\varepsilon = \frac{1}{2}$ the corresponding $\delta = \delta_1(z, W) > 0$, $c = c_1(z, W) \in (0,1)$. Further find to $\varepsilon = c_1(z, W)$ the corresponding $\delta = \delta_2(z, W) > 0$, $c = c_2(z, W) \in (0,1)$. Put

$$r(z, W^\star) = \frac{1}{2}\left[\min\left(\delta_1(z, W), \delta_2(z, W), c_1(z, W), c_2(z, W)\right)\right]^2.$$

Now let $V_x^\star$ be an $s^\star$-neighbourhood of x, $V_y^\star$ be an $s^\star$-neighbourhood of y such that

$$0 < \|y - x\| \leq \min(r(x, V_x^\star), r(y, V_y^\star)).$$

We shall write $\delta_{1,x}$ instead of $\delta_1(x, V_x)$ etc. We may suppose without loss of generality that $c_{1,x} \leq c_{1,y}$. Put

$$\xi = x + \frac{1}{c_{2,x}}(y - x).$$

Then

$$\begin{aligned} y - x &= c_{2,x}(\xi - x), \\ y - \xi &= (1 - c_{2,x})(x - \xi). \end{aligned}$$

Since $\|y - x\| = c_{2,x}\|\xi - x\| \leq r(x, V_x^\star) < c_{2,x} \cdot \delta_{2,x}$, it follows that $0 < \|\xi - x\| < \delta_{2,x}$. Hence there exists a ball $U(\eta, r)$ for which

$$\begin{aligned} U(\eta, r) &\subset U(\xi, (1 - c_{2,x})\|\xi - x\|) \cap V_x, \\ r &\geq (1 - c_{1,x})\|\eta - x\|. \end{aligned} \tag{5}$$

Furthermore,

$$\begin{aligned} \|\eta - y\| \leq \|\eta - \xi\| + \|\xi - y\| < 2\|x - \xi\| &= \frac{2}{c_{2,x}}\|y - x\| \\ &\leq \frac{2}{c_{2,x}}\sqrt{r(x, V_x^\star) \cdot r(y, V_y^\star)} \leq \frac{2}{c_{2,x}} \cdot \frac{1}{2} c_{2,x} \cdot \delta_{1,y} = \delta_{1,y}. \end{aligned}$$

This implies that there exists a ball $G = U(\eta', r')$ for which

$$\begin{aligned} U(\eta', r') &\subset U(\eta, (1 - c_{1,y})\|\eta - y\|) \cap V_y, \\ r' &\geq \frac{1}{2}\|\eta' - y\|. \end{aligned} \tag{6}$$

Lemma 2 yields (consider $u = y - \xi$, $v = \eta - \xi$, $\alpha u = x - \xi$)

$$\|\eta - y\| < \|\eta - x\|,$$

so

$$r \geq (1 - c_{1,x})\|\eta - x\| > (1 - c_{1,y})\|\eta - y\|,$$

which means that

$$U\big(\eta, (1 - c_{1,y})\|\eta - y\|\big) \subset U(\eta, r). \tag{7}$$

Combining (5), (6), (7) we obtain that

$$G \subset U(\eta, r) \cap V_y \subset V_x \cap V_y.$$

The completeness of Q implies that G is of second category, consequently

$$G \cap V_x^\star \cap V_y^\star \neq \emptyset. \quad \blacksquare$$

COROLLARY. *Any $s^\star$-continuous function on a Hilbert space Q is in the first class of Baire.*

5. THE CLASSES OF ZAHORSKI

For a real-valued function f defined on $\mathbf{R}$, the *associated sets of f* are all sets of the form $\{x : f(x) < \alpha\}$ or $\{x : f(x) > \alpha\}$. It is well-known that f is in the first class of Baire ($\mathcal{B}_1$) if and only if every associated set of f is of type F_σ. Z. Zahorski ([Z]) considered a hierarchy $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_5$ of subclasses of $\mathcal{B}_1$. Each of these classes is defined in terms of associated sets: f belongs to the *class $\mathcal{M}_i$ of Zahorski* iff every associated set of f belongs to the class M_i ($i = 0, 1, \ldots, 5$) where M_i is a certain family of F_σ sets.

In what follows, λ denotes the Lebesgue measure on $\mathbf{R}$.

Let $E \subset \mathbf{R}$ be a nonempty set of type F_σ. We say that E belongs to the *class*

M_0 if $E \cap I$ is infinite for any closed interval I which intersects E;

M_1 if $E \cap I$ is uncountable for any closed interval I which intersects E;

M_2 if $\lambda(E \cap I) > 0$ for any closed interval I which intersects E;

M_3 if

$$\lim_{n\to\infty} \frac{\lambda(I_n)}{\operatorname{dist}(x, I_n)} = 0$$

whenever $x \in E$ and $\{I_n\}$ is a sequence of closed intervals not containing x such that $\lambda(E \cap I_n) = 0$ for each n and $\lim_{n\to\infty}(\operatorname{diam}(\{x\} \cup I_n)) = 0$;

M_4 if there exists a sequence $\{H_n\}$ of closed sets and a sequence $\{\vartheta_n\}$ of positive numbers such that $E = \bigcup H_n$ and for each $x \in H_n$ and each $\gamma > 0$ there exists $\delta > 0$ such that whenever $h, h_1 \in \mathbf{R}$ satisfy $hh_1 > 0$, $h/h_1 \leq \gamma$, $|h + h_1| < \delta$, then

$$\frac{\lambda(E \cap (x + h, x + h + h_1))}{h_1} > \vartheta_n;$$

M_5 if every point $x \in E$ is a point of density of E, i.e. $\lim_{h\to 0} \dfrac{\lambda(E \cap (x, x + h))}{h} = 1.$

The empty set is considered to belong to each of these classes.

Z. Zahorski proved that $\mathcal{M}_0 = \mathcal{M}_1 = \mathcal{DB}_1$ (the collection of all functions in the first class of Baire which have the Darboux property). He also demonstrated that the class Δ of derivatives is contained in $\mathcal{M}_3$ and the class $b\Delta$ of bounded derivatives is contained in $\mathcal{M}_4$. Moreover, $\mathcal{M}_5$ is equal to the class $\mathcal{A}$ of approximately continuous functions.

For further information concerning the classes of Zahorski see [**B**].

Let $\mathcal{C}(X,\tau)$ denote the family of all real-valued functions on X which are continuous relative to the topology τ in the domain space and the Euclidean topology in the range. In the sequel we shall establish the relations $\mathcal{C}(\mathbf{R},p^\star) \subset \mathcal{M}_3$, $\mathcal{C}(\mathbf{R},s^\star) \subset \mathcal{M}_2$, $\mathcal{C}(\mathbf{R},p^\star) \not\subset \mathcal{M}_4$, $\mathcal{C}(\mathbf{R},s^\star) \not\subset \mathcal{M}_3$. (Recall that $\mathcal{C}(\mathbf{R},p^\star) = \mathcal{C}(\mathbf{R},p) \subset \mathcal{B}_1$, $\mathcal{C}(\mathbf{R},s^\star) = \mathcal{C}(\mathbf{R},s) \subset \mathcal{B}_1$.)

We shall use the following easy fact.

LEMMA 3. *A set $E \subset \mathbf{R}$ belongs to M_3 if and only if E belongs to M_2 and, moreover, E has porosity 0 at any point $x \in E$.*

PROPOSITION 4. $\mathcal{C}(\mathbf{R},p^\star) \cup \mathcal{C}(\mathbf{R},s^\star) \subset \mathcal{M}_2$.

PROOF: Suppose that f is a $p^\star$-continuous function. Let E be an associated set of f. Since f belongs to $\mathcal{B}_1$, E is of type F_σ. In light of the fact that E is p-open, applying the condition $(C_p{}')$ of Proposition 2a′ we see that E belongs to M_2.

The arguments for an $s^\star$-continuous function are similar. ∎

COROLLARY. (a) (cf. [**PBW**]) *Any $p^\star$-continuous function on $\mathbf{R}$ has the Darboux property.*

(b) *Any $s^\star$-continuous function on $\mathbf{R}$ has the Darboux property.*

PROPOSITION 5. $\mathcal{C}(\mathbf{R},p^\star) \subset \mathcal{M}_3$.

PROOF: It is sufficient to show that any p-open set E of type F_σ belongs to M_3. If $x \in E$, then $\mathbf{R} \setminus E$ is superporous at x by Proposition 1a. It follows that E has porosity 0 at x, so Lemma 3 together with Proposition 4 implies the result. ∎

PROPOSITION 6. *There exists a bounded $s^\star$-continuous function f which does not belong to $\mathcal{M}_3$.*

PROOF: Let $B \subset \mathbf{R}$ be the set constructed in Example 2. Then $E = \mathbf{R} \setminus B$ is an s-open set of type F_σ. Observe that $\pi(E,0) > 0$. Since the topology s has the Lusin-Menchoff property (Theorem 1b), by Theorem A there exists an $s^\star$-continuous function f such that $E = \{\, x : f(x) > 0 \,\}$. It is enough to show that E is not in M_3. But this follows from Lemma 3 since E is porous at 0. ∎

PROPOSITION 7. *There exists a bounded $p^\star$-continuous function f which does not belong to $\mathcal{M}_4$.*

PROOF: We shall construct a p-open set $E \subset \mathbf{R}$ of type F_σ which does not belong to M_4.

For $k = 0,1,2,\ldots$ put

$$I_k = \bigcup_{i=0}^{2^k-1} I_{k,i} \qquad \text{where} \quad I_{k,i} = \left(\frac{1}{2} + \frac{i}{2^{k+1}}, \frac{1}{2} + \frac{i}{2^{k+1}} + \frac{1}{4^{k+1}}\right).$$

Finally, define

$$E = (-\infty, 0] \cup \bigcup_{j=0}^{\infty} E_j.$$

Assume that E is in M_4. Then there exist sequences $\{H_n\}$, $\{\vartheta_n\}$ with properties described in the definition of the class M_4. Let n be an index for which $0 \in H_n$. Take a natural number m such that $1/2^{m-1} \le \vartheta_n$ and put $\gamma = 2^m - 1$. Find the corresponding $\delta = \delta(0, \gamma) > 0$ and choose a nonnegative integer j for which $1/2^j < \delta$. Then for $h = 1/2^j - 1/2^{j+m}$, $h_1 = 1/2^{j+m}$ we have $h/h_1 = \gamma$, $h + h_1 < \delta$. However,

$$\begin{aligned}\frac{\lambda(E \cap (h, h+h_1))}{h_1} &= 2^{j+m}\,\lambda\Big(E_j \cap \Big(\frac{1}{2^j} - \frac{1}{2^{j+m}}, \frac{1}{2^j}\Big)\Big) = 2^m\,\lambda\Big(E_0 \cap \Big(1 - \frac{1}{2^m}, 1\Big)\Big) \\ &= 2^m\,\lambda\Big(\bigcup_{k=m-1}^{\infty} \Big(I_k \cap \Big(1 - \frac{1}{2^m}, 1\Big)\Big)\Big) \le 2^m \sum_{k=m-1}^{\infty} 2^{k-(m-1)} \frac{1}{4^{k+1}} \\ &= 2^m \frac{1}{2^{2m-1}} = \frac{1}{2^{m-1}} \le \vartheta_n,\end{aligned}$$

which is a contradiction.

To prove that E is p-open, it suffices to show that the following modification of the condition $(C_p{}')$ from Proposition 2a′ is satisfied (on $\mathbf{R}$ it is customary to deal with intervals rather than balls—cf. [**Z3**]):

For any $c \in (0,1)$ there is $\delta > 0$ and $\varepsilon \in (0,1)$ such that whenever $0 < x < \delta$, then there exists an interval $J \subset E \cap (x - cx, x)$ for which $\lambda(J) \ge \varepsilon x$, and an interval $J' \subset E \cap (-x, -x + cx)$ for which $\lambda(J') \ge \varepsilon x$.

Choose $c \in (0,1)$. Put $\delta = 1$, $\varepsilon = c^2/256$. Fix $x \in (0,1)$. Let k be the integer for which $c^* = 1/2^k \in [c/2, c)$, and let j be the integer for which $y = 2^j x \in [\frac{1}{2}, 1)$. Observe that $\lambda((y - c^* y, y)) = c^* y \ge 1/2^{k+1}$. We shall distinguish two cases.

1. Assume that $\frac{1}{2} \le y - (c^*/2)y$. Then $\lambda\big((y - c^* y, y) \cap (\frac{1}{2}, 1)\big) \ge 1/2^{k+2}$. It follows that for some $i \in \{0, 1, \ldots, 2^{k+2} - 1\}$ we have

$$\Big(\frac{1}{2} + \frac{i}{2^{k+3}}, \frac{1}{2} + \frac{i+1}{2^{k+3}}\Big) \subset (y - c^* y, y) \cap (\tfrac{1}{2}, 1),$$

consequently $I_{k+2,i} \subset E_0 \cap (y - c^* y, y)$. Putting $J = (1/2^j) \cdot I_{k+2,i}$, we obtain

$$J \subset \frac{1}{2^j} \cdot [E_0 \cap (y - c^* y, y)] = E_j \cap (x - c^* x, x) \subset E \cap (x - cx, x),$$

$$\lambda(J) = \frac{1}{2^j} \cdot \frac{1}{4^{k+3}} > \frac{1}{4^{k+3}}\, x = \frac{1}{64} \cdot \Big(\frac{1}{2^k}\Big)^2 x \ge \frac{c^2}{256}\, x = \varepsilon x.$$

2. Assume that $y - (c^*/2)y < \frac{1}{2} \leq y$. Then $\lambda\big((y - c^*y,\, y) \cap (0, \frac{1}{2})\big) \geq 1/2^{k+2}$, hence

$$\frac{1}{2} \cdot I_{k,2^k-1} \subset \frac{1}{2} \cdot \left[E_0 \cap \left(\frac{1}{2} + \frac{2^k - 1}{2^{k+1}}, 1\right)\right] = E_1 \cap \left(\frac{1}{2} - \frac{1}{2^{k+2}}, \frac{1}{2}\right) \subset E_1 \cap (y - c^*y,\, y).$$

Putting $J = (1/2^{j+1}) \cdot I_{k,2^k-1}$, we have

$$J \subset \frac{1}{2^j} \cdot [\, E_1 \cap (y - c^*y,\, y)\,] = E_{j+1} \cap (x - c^*x,\, x) \subset E \cap (x - cx,\, x),$$
$$\lambda(J) = \frac{1}{2^{j+1}} \cdot \frac{1}{4^{k+1}} > \frac{1}{2 \cdot 4^{k+1}}\, x = \frac{1}{8} \cdot \left(\frac{1}{2^k}\right)^2 x \geq \frac{c^2}{32}\, x > \varepsilon x.$$

Since the existence of the interval J' is obvious, we have demonstrated that E is p-open.

By Theorem 1a, the topology p has the Lusin-Menchoff property. Theorem A guarantees that there exists a bounded p^*-continuous function f for which $E = \{\, x : f(x) > 0\,\}$. On the other hand, we have shown that E is not in M_4, therefore f does not belong to the class $\mathcal{M}_4$ and the proof is complete. ∎

Let us remark that Proposition 7 slightly improves a result of W. Wilczyński and V. Aversa ([**WA**, Theorem 3]). They constructed a bounded $\mathcal{I}$-approximately continuous function which is not a derivative.

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