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Pseudo–Orbit Shadowing on the Unit Interval

I. Introduction

Let $f : X \to X$ be a continuous function where $X = [0,1]$ has the usual topology. Then $(X, f, \mathbb{N})$ is a *dynamical system* where $f^n, n \in \mathbb{N}$, is the n-fold composition of f. The *orbit* of $x \in X$ is the sequence $\{x, f(x), \ldots, f^n(x), \ldots\}$, $n \in \mathbb{N}$. A pseudo-orbit is defined as follows: Given $\delta > 0$, a δ - *pseudo-orbit* is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $d(f(x_i),\ x_{i+1}) \leq \delta \ \ \forall\, i \in \mathbb{N}$.

A dynamical system with noise will generate a pseudo-orbit, an example being an orbit generated by a computer. Consequently, the relationship between actual orbits and pseudo-orbits raises interesting questions. To wit, when are pseudo-orbits closely followed (i.e. shadowed) by actual orbits? A function f has the *shadowing property* if $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that given a δ - pseudo-orbit, $\{x_n\}_{n=0}^{\infty}$, there is an $x \in X$ which satisfies $d(x_n,\ f^n(x)) < \varepsilon \ \ \forall\, n \in \mathbb{N}$. In this case, we say the δ - pseudo orbit is ε - *shadowed* by the actual orbit.

Coven, Kan, and Yorke considered the shadowing property in the family of tent maps[1]: diffeomorphisms have also been studied. In this paper we consider functions that are increasing (i.e. nondecreasing) and continuous on the unit interval. Let $\mathcal{F}$ be the set of all interior fixed points of f, and take $\mathcal{C} = \{x \in \mathcal{F} : \forall\, \varepsilon > 0 \ \ \exists\, y, z \in (x - \varepsilon,\ x + \varepsilon)$ such that $f(y) < y$ and $f(z) > z\}$. Our main result (Theorem 8) is that an increasing continuous function $f : [0,1] \to [0,1]$ will have the shadowing property if and only if $\mathcal{F} = \mathcal{C}$.

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II. PRELIMINARIES

Let $x \in X$ be a fixed point of f, which throughout the paper will denote an increasing (i.e. nondecreasing) continuous function on $[0,1]$, and let $I = (x-\alpha,\ x+\alpha)$, $\alpha > 0$. If $\exists\ \alpha > 0$ such that all actual orbits $\{a_n\}_{n=0}^{\infty}$ (i.e., $a_n = f^n(a_0)$, $a_0 \in [0,1]$) in I converge to x, then we call x an *attracting fixed point.* On the other hand, x is a *repelling fixed point* if $\exists\ \alpha > 0$ such that all actual orbits $\{a_n\}_{n=0}^{\infty}$ that start in $I \setminus \{x\}$ eventually leave I. This leads us to the following propositions.

Proposition 1. *The interior fixed point x is attracting if and only if $\exists\ \varepsilon > 0$ such that $f(t) > t$ for $t \in (x-\varepsilon,\ x)$ and $f(t) < t$ for $t \in (x,\ x+\varepsilon)$.*

PROOF. $\Longrightarrow$ Suppose $\forall\ \varepsilon > 0\ \ \exists\ t \in (x-\varepsilon,\ x)$ with $f(t) \leq t$. (The other case is proved similarly.)

Case 1. Suppose $\exists\ t \in (x-\varepsilon,\ x)$ such that $f(t) = t$. Let the actual orbit $\{a_n\}_{n=0}^{\infty}$ have initial value t. Then $\{a_n\}_{n=0}^{\infty}$ converges to t instead of x.

Case 2. If there is a $t \in (x-\varepsilon,\ x)$ such that $f(t) < t$, then, since f is increasing, an orbit beginning at t will have no elements greater than t. In particular, the orbit will not converge to x.

$\Longleftarrow$ Since $\exists\ \varepsilon > 0$ such that $f(t) > t\ \ \forall\ t \in (x-\varepsilon,\ x)$, $\{a_n\}_{n=0}^{\infty}$ starting in $(x-\varepsilon,\ x)$ is increasing and $a_0 < x$. If $a_n < x$, $n \in I\!N$, then $a_{n+1} = f(a_n) \leq f(x) = x$ since f is increasing. Thus x is an upper bound of $\{a_n\}_{n=0}^{\infty}$, and as is easily seen, $sup\ a_n = x$. □

Proposition 2. *The interior fixed point x is repelling if and only if $\exists\ \varepsilon > 0$ such that $f(t) < t\ \ \forall\ t \in (x-\varepsilon,\ x)$ and $f(t) > t\ \ \forall\ t \in (x,\ x+\varepsilon)$.*

PROOF. $\Longrightarrow$ Clear by using the contrapositive.

$\Longleftarrow$ Let $\alpha = \varepsilon/2$. Choose $\{a_n\}_{n=0}^{\infty}$ to be any actual orbit starting in $(x-\varepsilon/2,\ x)$. Set $d = min\{|f(t)-t| :\ t \in [x-3\varepsilon/4,\ a_1]\}$. Let N be the smallest integer greater than $(a_1 - (x-3\varepsilon/4))/d$. Since $a_n - a_{n-1} \geq d\ \forall n$, $a_N \notin (x-\varepsilon/2,\ x)$. □

III. THE SHADOWING PROPERTY

We begin by considering the simplest case in which f has no interior fixed points. This is then generalized to one, finitely many, and an arbitrary number of fixed points.

Proposition 3. *If f satisfies $f(t) > t \ \forall\, t \in (0,1)$, then f has the shadowing property.*

PROOF. Given $\varepsilon > 0$, let $d = min\{|f(t) - t| \ : t \in [\varepsilon/2,\ 1 - \varepsilon/2]\}$ Hence d is the minimum difference between any two consecutive elements of any actual orbit contained in $[\varepsilon/2,\ 1-\varepsilon/2]$. Let $\eta = min\{\varepsilon/2,\ d/2\}$, and take N_p to be the smallest integer greater than $(1-\varepsilon)/(d-\eta)$. Since f is uniformly continuous on $[0,1]$, $\exists\, \delta_{N_p}$, $0 < \delta_{N_p} < \varepsilon/2(N_p+1)$, such that $|f(t) - f(x)| < \varepsilon/2(N_p+1)$ whenever $|t - x| < \delta_{N_p}$ $\forall\, t, x \in [0,1]$. Regressing, there exists $\delta_{k-1} < \delta_k$ such that whenever $|t-x| < \delta_{k-1}$, then $|f(t)-f(x)| < \delta_k$, $(k = 2, 3, \ldots, N_p)$, $\forall\, t, x \in [0,1]$. Choose $\delta > 0$ such that $\delta < min\{\delta_1,\ \eta,\ \varepsilon/2(N_p+1)\}$. Since $\delta < \eta \le d/2$ and $f(x) - x \ge d$ on $[\varepsilon/2,\ 1-\varepsilon/2]$, if x_k and x_{k+1} are two consecutive points of a δ - pseudo-orbit in $[\varepsilon/2,\ 1-\varepsilon/2]$, then $x_{k+1} - x_k > d - \eta$. Hence N_p is an upper bound for the number of iterations needed for any δ - pseudo-orbit to travel from $\varepsilon/2$ to $1 - \varepsilon/2$.

Case 1. $f(0) = 0$.

(A) The δ - pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ has an infinite number of elements in $[0,\varepsilon)$: If there is an element of $\{x_n\}_{n=0}^{\infty}$ outside of $[0,\varepsilon)$, call it x_s, then since $\{x_n\}_{n=0}^{\infty}$ is increasing on $[\varepsilon/2,\ 1-\varepsilon/2]$, $x_{s+k} \notin [0,\varepsilon)$ $\forall\, k \in \mathbb{N}$. Thus if an infinite number of elements of $\{x_n\}_{n=0}^{\infty}$ belong to $[0,\varepsilon)$, then $x_n \in [0,\varepsilon)$ $\forall\, n \in \mathbb{N}$. Let $\{a_n\}_{n=0}^{\infty}$ be an actual orbit with initial value equal to 0. Since 0 is a fixed point, $a_n = 0$ $\forall\, n \in \mathbb{N}$. Hence $\{a_n\}_{n=0}^{\infty}$ ε - shadows the pseudo-orbit $\{x_n\}_{n=0}^{\infty}$.

(B) The δ - pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ has only a finite number of elements in $[0,\varepsilon)$: Let x_r be the last element of $\{x_n\}_{n=0}^{\infty}$ in $[0,\varepsilon)$. Set $a_r = x_r$ and set $a_{r-n} \in f^{-n}(a_r)$ for $n = 1, 2, \ldots, r$). Since $a_r \in [0,\varepsilon)$, $a_0, a_1, \ldots, a_{r-1} \in [0,\varepsilon)$. Thus $|a_n - x_n| < \varepsilon$ for $n = 0, 1, \ldots, r$.

Now, $|f(x_r) - x_{r+1}| < \delta < \delta_1$. Hence $|f^2(x_r) - f(x_{r+1})| < \delta_2$, and $|f(x_{r+1}) - x_{r+2}| < \delta < \delta_2$. Continuing, we finish with $|f^{N_p+1}(x_r) - f^{N_p}(x_{r+1})| < \varepsilon/2(N_p+1)$, $|f^{N_p}(x_{r+1}) - f^{N_p-1}(x_{r+2})| < \varepsilon/2(N_p+1)$, $\ldots$, $|f(x_{N_p}) - x_{N_p+1}| < \delta < \varepsilon/2(N_p+1)$. We see $|f^{N_p+1}(x_r) -$

$x_{N_p+1}| < (\varepsilon/2(N_p+1))(N_p+1) = \varepsilon/2$. Since $\delta_k < \delta_{k+1} < \varepsilon/2(N_p+1)$, for $k = 1, 2, \ldots, N_p$, $|f^k(x_r) - x_{r+k}| < \varepsilon/2$. Hence $|a_{r+k} - x_{r+k}| < \varepsilon/2$ for $k = 1, 2, \ldots, N_p + 1$. Because $N_p + 1$ iterations have taken place since a_r, $a_{r+N_p+1} \in (1 - \varepsilon/2,\ 1]$. Thus $x_{r+N_p+1} \in (1 - \varepsilon,\ 1]$. It follows from $\delta < \varepsilon/2$ and $\delta < d/2$, that $x_{r+N_p+j} \in (1 - \varepsilon,\ 1]\ \ \forall\, j \in \mathbb{N}$. Since $\{a_n\}_{n=0}^{\infty}$ is increasing for all n, $a_{r+N_p+j} \in (1-\varepsilon,\ 1]\ \forall\, j \in \mathbb{N}$. Hence $|a_{r+N_p+j} - x_{r+N_p+j}| < \varepsilon\ \ \forall\, j \in \mathbb{N}$. Therefore $|a_n - x_n| < \varepsilon\ \ \forall\, n \in \mathbb{N}$.

(C) The pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ has no elements in $[0, \varepsilon)$: Let $a_0 = x_0$. The rest is similar to (B).

Case 2. $f(0) > 0$: Choose $0 < \varepsilon_1 < min\{\varepsilon,\ f(0)/2\}$. Now using ε_1 instead of ε, choose $\delta > 0$ with δ fulfilling all the previous conditions. Since $\varepsilon_1 < f(0)/2$ and $\delta < \varepsilon_1/2$, any actual or δ - pseudo orbit starting in $[0, \varepsilon_1)$ has only one element in $[0, \varepsilon_1)$. The rest of the argument is similar to Case 1. □

A similar argument proves

Proposition 4. *If $f(t) < t\ \ \forall\, t \in (0,1)$, then f has the shadowing property.*

Next, we consider the cases when f has one interior fixed point, being either attracting or repelling.

Proposition 5. *If f has one interior fixed point x which is attracting, then f has the shadowing property.*

PROOF. If x is the interior fixed point, then Propositions 3 & 4 can be applied to the intervals $[0, x]$ and $[x, 1]$ respectively. Since the pseudo orbit can travel past the fixed point, we need only choose δ small enough that the pseudo orbit stays close to the fixed point. □

Proposition 6. *If f has one interior fixed point x which is repelling, then f has the shadowing property.*

This proof is similar to that of the previous result except for the case when a pseudo-orbit originates in a small neighborhood about the fixed point. Either the pseudo-orbit stays in the neighborhood indefinitely in which case take $a_0 = x$, or else it eventually leaves the neighborhood which then commits the pseudo-orbit to increase towards 1 or decrease towards 0. □

Now that we have dealt with a single interior fixed point, we move to a finite number of attracting or repelling interior fixed points. Here, f will still have the shadowing property, since we can trap the pseudo-orbit by one of the fixed points as was done in the previous arguments. We state this as

Proposition 7. *If f has a finite number of interior fixed points, each one either attracting or repelling, then f has the shadowing property.*

PROOF. Given $\varepsilon > 0$, let $s_1, s_2, \ldots, s_{N-1}$ be the interior fixed points, and set s_0 and s_N equal to 0 and 1 respectively. Let $I_1 = (s_0, s_1)$, $I_2 = (s_1, s_2), \ldots,$ $I_N = (s_{N-1}, s_N)$, and $\sigma = min\{|I_1|, |I_2|, \ldots, |I_N|\}$. Also, let $\rho = min\{\varepsilon/2, \sigma/4\}$ and $d_1 = min\{|f(x) - x|, x \in [s_0 + \rho/2, s_1 - \rho/2]\}$, likewise $d_2, d_3, \ldots, d_N$. Take $d = min\{d_1, d_2, \ldots, d_N\}$, and set $\eta = min\{\rho/2, d/2\}$. Let N_{p_1} be the smallest integer greater than $(s_1 - \rho)/(d - \eta)$, likewise $N_{p_2} \geq (s_2 - s_1 - \rho)/(d - \eta), \ldots, N_{p_N} \geq (s_N - s_{N-1} - \rho)/(d - \eta)$. Take $N_p = max\{N_{p_1}, N_{p_2}, \ldots, N_{p_N}\}$. Choose $\delta_{N_p}, \delta_{N_p - 1}, \ldots, \delta_1$ as in the proof of Proposition 3. Choose δ such that $0 < \delta < min\{\delta_1, \eta, \varepsilon/(N_p + 1)\}$. Since $\delta < \eta$, every δ - pseudo-orbit will be contained in $(s_{j-1} - \rho, s_j + \rho)$ for some $j = 1, 2, \ldots, n$. Without loss of generality, let s_j be attracting and s_{j-1} be repelling. The argument then follows as it did for Proposition 5. □

Recalling the definition of $\mathcal{C}$ and $\mathcal{F}$ from the Introduction, we now show that f has the shadowing property if $\mathcal{F} = \mathcal{C}$ by isolating the fixed points in a finite number of small intervals, and then treating the intervals like fixed points. We also show the condition $\mathcal{F} = \mathcal{C}$ is necessary as well as sufficient for f to have the shadowing property, resulting in the following theorem.

Theorem 8. *Let f be an increasing continuous function on $[0,1]$, then f has the shadowing property if and only if $\mathcal{F} = \mathcal{C}$.*

PROOF. $\Longrightarrow$ Suppose $\mathcal{C} \neq \mathcal{F}$. Then there exists a fixed point x and $\alpha > 0$ such that either $f(t) \geq t$ or $f(t) \leq t \ \ \forall\, t \in (x - \alpha,\ x + \alpha)$; assume the former. Let $\varepsilon = \alpha/4$ and $\delta > 0$ be given. We can find a δ - pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ that starts at $x - 3\alpha/4$ and such that $x_0 < x_1 < \cdots < x_m$ and $x_m > x + \alpha/4$ for some $m \in I\!N$. Any actual orbit $\{a_n\}_{n=0}^{\infty}$ in order to ε - shadow $\{x_n\}_{n=0}^{\infty}$ must start in $(x - \alpha,\ x - \alpha/2)$. Since $a_n < x \ \ \forall\, n \in I\!N$, $|a_m - x_m| > \alpha/4 = \varepsilon$.

$\Longleftarrow$ Let $\varepsilon > 0$ be given. Call an interval (a,b) *attracting* if $f(a) > a$ and $f(b) < b$, and *repelling* if $f(a) < a$ and $f(b) > b$. Let $\mathcal{P} = \{0, a_1, a_2, \ldots, a_N, 1\}$ be a partition of $[0,1]$ with $\|\mathcal{P}\| < \varepsilon/5$. Since $\mathcal{F} = \mathcal{C}$, we may choose $\mathcal{P}$ so that $a_i \notin \mathcal{F} \ \ \forall\, i = 1,2,\ldots,N$ and also so that each interval (a_i, a_{i+1}) containing a fixed point is either attracting or repelling.

Now, let $\mathcal{J}$ be the collection of all intervals of the form (a_i, a_{i+1}) which contain a fixed point. Since $a_i \notin \mathcal{F} \ \ \forall\, i = 1,2,\ldots,N$, we can shorten each interval of $\mathcal{J}$ slightly and throw in $I_1 = [0, a_1)$ and $I_M = (a_N, 1]$ to form a finite collection of disjoint open intervals, numbered left to right, $I_1, I_2, \ldots, I_M$ which satisfy:

1) $\mathcal{F} \subset \bigcup_{k=1}^{M} I_k$.

2) Each $I_k,\ k = 2,3,\ldots,M-1$, is attracting or repelling.

3) $|I_k| < \varepsilon/5,\ k = 1,2,\ldots,M$.

4) There is a positive distance between each pair of intervals.

The attracting and repelling intervals $I_2, I_3, \ldots, I_{M-1}$ by virtue of their construction behave like attracting and repelling fixed points. Hence we can treat the intervals $I_2, I_3, \ldots, I_{M-1}$ as we did the points $s_1, s_2, \ldots, s_{N-1}$ in Proposition 7. With this in mind, set σ equal to the minimum distance between I_m and I_{m+1}, $1 \leq m < M$. Let $\rho = min\{\varepsilon/5,\ \ \sigma/5\}$. Choose $0 < \delta < \rho$ so that δ fulfills the conditions from the proof of Proposition 7. If I_n, $(n = 2,3,\ldots,M-1)$, is attracting, then once a δ - pseudo-orbit has an element in $(I_n - \rho,\ I_n + \rho)$, all following elements of the δ - pseudo-orbit are also in $(I_n - 2\rho,\ I_n + 2\rho)$ which has length less than ε. The rest of the proof is similar to the proof of Proposition 7. □

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Example 9. Since the standard Cantor function defined on the unit interval is increasing, continuous, and is easily shown to have exactly one interior fixed point, the Cantor function has the shadowing property.

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