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An Analytic Study of Functions defined on Self–Similar Fractals

1. Introduction

Let E be an s-set, that is, a subset of the Euclidean n-space $\mathbb{R}^n$ which is measurable with respect to the s-dimensional Hausdorff measure H^s and for which $0 < H^s(E) < \infty$. In the Mandelbrot's terminology [6], a fractal is a s-set for which s is fractional, or s is integer and its geometric properties are completely opposites to the properties of the nice s-dimensional surfaces. The aim of this paper is the construction of a Fourier Analysis on the self-similar s-sets, that is s-sets which are a finite union of disjoint subsets, each of them similar to the whole set.

In the second section of this paper we present some definitions and results about Hausdorff measures and self-similar sets, and notations.

Given $E \subset \mathbb{R}^n$, a self-similar s-set, in section 3 we define on E a system of functions $\Phi \subset L^\infty(E, H^s)$ which will be orthonormal in the Hilbert space $L^2(E, H^s)$. Later we evaluate the Dirichlet kernel associated to Φ and we get the pointwise convergence of the Fourier series, with respect to Φ, for functions $f \in L^1(E, H^s)$, which give us immediately the L^p-completeness of Φ, for $p \geq 1$. Finally we study in this section the convergence to zero of the Fourier coefficients of the functions $f \in L^1(E, H^s)$ and the convergence in L^p-norm of the Fourier series, with respect to Φ, of the functions $f \in L^p(E, H^s)$.

Finally, in section 4, we give some examples where we obtain the Fourier series of some functions and the functions associated to some Fourier coefficients.

The possibility of this theory was suggested by Miguel de Guzmán in [3]. I wish to acknowledge here his kind and generous advice.

2. Preliminaries

Given a subset $E \subset \mathbb{R}^n$, for s, $0 \leq s \leq n$, we define:

$$H^s(E) = \liminf_{\delta \to 0^+} \{\sum_{i=1}^{\infty} d(A_i)^s \ : \ E \subset \bigcup_{i=1}^{\infty} A_i \ , \ 0 < d(A_i) \leq \delta\}$$

where $d(A)$ denotes the diameter of the set A. For each s, $0 \leq s \leq n$, the application H^s is an outer measure on $\mathbb{R}^n$ which we call **Hausdorff s-dimensional outer measure**. The restriction of H^s

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to the σ-field of H^s-measurable sets is called **Hausdorff s-dimensional measure.** This measure H^s is Borel regular. We shall say that E is an **s-set** if E is H^s-measurable and $0 < H^s(E) < \infty$.

Given $E \subset \mathbb{R}^n$ there is a unique number s, $0 \leq s \leq n$, such that:

$$H^t(E) = \infty \ , \quad \text{if} \quad t < s$$

$$H^t(E) = 0 \ , \quad \text{if} \quad t > s$$

which is called the **Hausdorff dimension** of E, and we shall write $s = dimE$.

More details about Hausdorff measures may be consulted in [2] or [8].

A wide and important class of s-sets is the one formed by the self-similar sets, which are defined as follows:

2.1. Definition

We shall say that a compact set $E \subset \mathbb{R}^n$ is **self-similar** if there is a finite family $\mathcal{S} = \{S_1, ..., S_\ell\}$ of similitudes ($S_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $|S_i(x) - S_i(y)| = r_i \cdot |x - y|$, $\forall x, y \in \mathbb{R}^n$, $0 < r_i < 1$, where r_i is called similitude ratio) and a number s, $0 \leq s \leq n$, verifying:

a) $E = \bigcup_{i=1}^{\ell} S_i(E)$

b) $0 < H^s(E) < \infty$

c) $H^s(S_i(E) \cap S_j(E)) = 0$, for $1 \leq i < j \leq \ell$

The number s which gives us the Hausdorff dimension of the self-similar set E is the unique positive number which verifies $\sum_{i=1}^{\ell} r_i^s = 1$ (see [2]). From the definition it follows immediately that if $\ell = 1$ the set E contains only one point, and the $dimE = 0$ and $H^0(E) = 1$. Therefore, we shall assume, in this paper, that $\ell \geq 2$.

2.2. Notation

Let ℓ be a fixed positive integer. For every positive integer k, we denote by $\mathcal{S}_k^\ell$ the set of all k-tuples formed using the first ℓ positive integers, that is:

$$\mathcal{S}_k^\ell = \{(i_1, ..., i_k) \ : \ 1 \leq i_j \leq \ell \ , \ 1 \leq j \leq k\}$$

and analogously $\mathcal{S}_\infty^\ell$ will denote the set of all the infinite sequences formed using the ℓ first positive integers:

$$\mathcal{S}_\infty^\ell = \{(i_1, ..., i_k, ...) \ : \ 1 \leq i_j \leq \ell \ , \ j \geq 1\}$$

If $\alpha = (i_1, ..., i_k) \in \mathcal{S}_k^\ell$ and $\beta = (j_1, ..., j_q) \in \mathcal{S}_q^\ell$, we define the **concatenation** of α and β by:

$$\alpha\beta = (i_1, ..., i_k, j_1, ..., j_q) \in \mathcal{S}_{k+q}^\ell$$

and for every p, $1 \leq p \leq k$, we denote by $\alpha[p]$ the p-tuple formed by the p first coordinates of α, that is:

$$\alpha[p] = (i_1, ..., i_p) \in \mathcal{S}_p^\ell$$

Moreover, if $k < q$, we shall say that $\alpha \subset \beta$ if $\alpha = \beta[k]$, that is, if the k-tuple α are the k first coordinates of β.

For every $k \geq 1$, we shall define in $\mathcal{S}_k^\ell$ the following natural order relation: Given $\alpha, \beta \in \mathcal{S}_k^\ell$, $\alpha = (i_1, ..., i_k)$ and $\beta = (j_1, ..., j_k)$, we shall say that $\alpha < \beta$ if there is h, $1 \leq h \leq k$, such that:

$$\begin{aligned} i_p &= j_p \ , \quad \text{if} \quad 1 \leq p < h \\ i_h &< j_h \end{aligned}$$

If $\alpha \in \mathcal{S}_k^\ell$ and $\beta \in \mathcal{S}_q^\ell$ with $k < q$, we shall say that $\alpha < \beta$.

With this relation we have ordered, in an only chain, all the elements of $\bigcup_{k\geq 1} \mathcal{S}_k^\ell$. We denote, for every $\alpha \in \mathcal{S}_k^\ell$ and $k \geq 1$, by α^- the preceding element to α in that chain, that is:

$$\alpha^- = \begin{cases} \max\{\beta \, : \, \beta \in \mathcal{S}_k^\ell \, , \, \beta < \alpha\} & ,\text{if } \alpha \neq (1,...,1) \\ (\ell,...,\ell) \in \mathcal{S}_{k-1}^\ell & ,\text{if } \alpha = (1,...,1) \end{cases}$$

2.3. Basic results about self-similar fractals

Let $E \subset I\!R^n$ be the self-similar set associated to the family of similitudes $\mathcal{S} = \{S_1, ..., S_\ell\}$, with ratios $\{r_1, ..., r_\ell\}$, and with Hausdorff dimension s ($\sum_{i=1}^\ell r_i^s = 1$). For every $k \geq 1$ and $\alpha = (i_1, ..., i_k) \in \mathcal{S}_k^\ell$ we denote:

$$E_\alpha = S_\alpha(E) = S_{i_1} \circ ... \circ S_{i_k}(E)$$

$$r_\alpha = r_{i_1} \cdot ... \cdot r_{i_k}$$

where r_α will be the similitude ratio of S_α. It is easy to show that:

(a) $\sum_{\alpha \in \mathcal{S}_k^\ell} r_\alpha^s = 1$, for all $k \geq 1$.

(b) $E = \bigcup_{\alpha \in \mathcal{S}_k^\ell} E_\alpha$, for all $k \geq 1$.

(c) $E_\alpha = \bigcup_{\beta \in \mathcal{S}_p^\ell} E_{\alpha\beta}$, for every $\alpha \in \mathcal{S}_k^\ell$, $k \geq 1$ and $p \geq 1$. In particular $E_\alpha = \bigcup_{i=1}^\ell E_{\alpha i}$.

(d) $H^s(E_\alpha \cap E_\beta) = 0$, for every $k \geq 1$, $\alpha, \beta \in \mathcal{S}_k^\ell$ and $\alpha \neq \beta$.

(e) If $\alpha \in \mathcal{S}_p^\ell$, $\beta \in \mathcal{S}_q^\ell$ and $p < q$, we have that:

  (e1) $E_\beta \subset E_\alpha$, if $\alpha \subset \beta$.

  (e2) $H^s(E_\alpha \cap E_\beta) = 0$, if $\alpha \not\subset \beta$.

(f) For every $k \geq 1$ and $\alpha \in \mathcal{S}_k^\ell$ we have, applying the homogeneity of the measure H^s, that:

  (f1) $H^s(E_\alpha) = r_\alpha^s \cdot H^s(E)$.

  (f2) $H^s(E_{\alpha\beta}) = r_\beta^s \cdot H^s(E_\alpha)$, for all $p \geq 1$ and $\beta \in \mathcal{S}_p^\ell$.

For more details about self-similar sets one can see [2] or [5].

Let $E \subset I\!R^n$ be the self-similar set, of dimension s, associated to the family of similitudes $\mathcal{S} = \{S_1, ..., S_\ell\}$ with ratios $\{r_1, ..., r_\ell\}$. We shall call **subsets of the generation** k, $k \geq 1$, to the elements of the following family:

$$\mathcal{E}_k = \{E_\alpha \, : \, \alpha \in \mathcal{S}_k^\ell\}$$

and we shall also consider the following families of sets:

$$\mathcal{A}_1 = \mathcal{E}_1 \cup \{E^j = \bigcup_{i=1}^{j} E_i \, : \, 1 < j < \ell\}$$

$$\mathcal{A}_k = \mathcal{E}_k \cup \{E_\alpha^j = \bigcup_{i=1}^{j} E_{\alpha i} \, : \, \alpha \in \mathcal{S}_{k-1}^\ell \, , \, 1 < j < \ell\} \quad , \quad \text{for all} \quad k > 1$$

It is easy to check that $\mathcal{B} = \bigcup_{k=1}^\infty \mathcal{A}_k$ is a **differentiation basis** for (E, H^s), being:

$$\mathcal{B}(x) = \{V \in \mathcal{B} \, : \, x \in V\} \quad , \quad \text{for all} \quad x \in E$$

This basis differentiates to $L^1(E, H^s)$ as show the next Theorem.

2.4. Theorem:

For every function $f \in L^1(E, H^s)$ we have that:

$$\lim_{\substack{d(U)\to 0\\ U\in\mathcal{B}(x)}} \frac{1}{H^s(U)} \cdot \int_U f(y) \cdot dH^s(y) = f(x)$$

for H^s-a.e. $x \in E$.

The proof of this Theorem can be found in [7]. This Theorem will be important to get the pointwise convergence results that we shall obtain later.

3. Fourier Analysis on Self-Similar Fractals

In this section $E \subset I\!R^n$ will be the self-similar fractal associated to the family of similitudes $\mathcal{S} = \{S_1, ..., S_\ell\}$, with ratios $\{r_1, ..., r_\ell\}$, and with Hausdorff dimension s ($\sum_{i=1}^{\ell} r_i^s = 1$). To simplify the computations we may suppose, without loosing generality, that $H^s(E) = 1$ and that $E_i \cap E_j = \emptyset$, for $1 \leq i < j \leq \ell$.

We are going to define on E a system of functions $\Phi \subset L^\infty(E, H^s)$ which will be orthonormal in $L^2(E, H^s)$. The appropriate computation of the Dirichlet kernel of this system, together with the differentiation Theorem 2.4, will allow us to show that the Fourier series (with respect to Φ) of every function of $L^1(E, H^s)$ converges at almost every point to the given function.

3.1. The system Φ of functions defined on E

We shall define a **function of the generation** -1 with support on the set E, which will be:

$$\gamma_{-1}(x) = 1 \quad , \text{ for all } x \in E$$

Associated to the set E we define $\ell - 1$ functions γ_0^h, $1 \leq h \leq \ell - 1$, which we call **functions of the generation** 0. They are defined as follows:

$$\gamma_0^h(x) = \begin{cases} C_h^{-\frac{1}{2}} & , \text{if } x \in \bigcup_{j=1}^{h} E_j \\ -C_h^{-\frac{1}{2}} \left(\sum_{i=1}^{h} r_i^s\right) r_{h+1}^{-s} & , \text{if } x \in E_{h+1} \\ 0 & , \text{otherwise} \end{cases}$$

and associated to every set E_α, $\alpha \in \mathcal{S}_k^\ell$, of the generation k, $k \geq 1$, we define $\ell - 1$ functions γ_α^h, $1 \leq h \leq \ell - 1$, which we call **functions of the generation** k (we shall have $\ell^k(\ell - 1)$ functions of the generation k) and they are:

$$\gamma_\alpha^h(x) = \begin{cases} C_h^{-\frac{1}{2}} r_\alpha^{-\frac{s}{2}} & , \text{if } x \in \bigcup_{j=1}^{h} E_{\alpha j} \\ -C_h^{-\frac{1}{2}} r_\alpha^{-\frac{s}{2}} \left(\sum_{i=1}^{h} r_i^s\right) r_{h+1}^{-s} & , \text{if } x \in E_{\alpha,h+1} \\ 0 & , \text{otherwise} \end{cases}$$

where for every h, $1 \leq h \leq \ell - 1$, the constant C_h is defined by:

$$C_h = \left(1 + r_{h+1}^{-s} \sum_{i=1}^{h} r_i^s\right) \sum_{i=1}^{h} r_i^s = r_{h+1}^{-s} \sum_{i=1}^{h} r_i^s \sum_{i=1}^{h+1} r_i^s \tag{1}$$

It is easy to see that:

$$\begin{aligned} support\ \gamma_{-1} &= E \\ support\ \gamma_0^h &= \bigcup_{j=1}^{h+1} E_j \subset E \quad , \ 1 \le h \le \ell-1 \\ support\ \gamma_\alpha^h &= \bigcup_{j=1}^{h+1} E_{\alpha j} \subset E_\alpha \quad , \ \alpha \in \mathcal{S}_k^\ell \ , \ k \ge 1 \ , \ 1 \le h \le \ell-1 \end{aligned}$$

and also that the system:

$$\Phi = \{\gamma_{-1}\} \cup \{\gamma_0^h \ : \ 1 \le h \le \ell-1\} \cup \{\gamma_\alpha^h \ : \ \alpha \in \mathcal{S}_k^\ell \ , \ k \ge 1 \ , \ 1 \le h \le \ell-1\}$$

verifies that $\Phi \subset L^\infty(E, H^s) \subset L^p(E, H^s)$, for every $p \ge 1$.

3.2. Remarks

(a) If $H^s(E) \neq 1$, we would have to divide the functions of Φ by $(H^s(E))^{\frac{1}{2}}$, and if the intersections of those sets were not empty (they are of H^s-measure zero) it would be enough to define the functions as zero on such intersections.

(b) If $\ell = 2$ then, associated to every set E_α of the generation k, $k \ge 1$, we shall have a unique function γ_α of the generation k (we shall also have a unique function of the generation zero).

(c) The interval $[0,1] \subset I\!R$ may be considered as a self-similar set of dimension 1 associated to the similitudes $\{S_1, S_2\}$ centered, respectively, in the points 0 and 1 and with ratios $r_1 = r_2 = \frac{1}{2}$. In this case the sets E_α, $\alpha \in \mathcal{S}_k^2$ and $k \ge 1$, are the dyadic partitions of $[0,1]$ and the system of functions defined on $E = [0,1]$ coincides (except in the values of the extremes of the intervals) with a system of functions defined by Haar in such interval and based in their dyadic partitions (see [1,pp.46]).

(d) If E is the classic Cantor set defined on the unit interval, then $dim E = \log 2/\log 3 = s$ and $H^s(E) = 1$. The classic Cantor set is the self-similar set associated to the homothecies $\{S_1, S_2\}$ centered, respectively, in the points 0 and 1 and both of ratio $r_1 = r_2 = \frac{1}{3}$. In this case $C_1 = 1$ and the system Φ associated to such Cantor set is:

$$\gamma_{-1}(x) = 1 \quad , \ \text{for all} \ x \in E$$

$$\gamma_0(x) = \begin{cases} 1 & , \text{if } x \in E_1 = E \cap [0, \frac{1}{3}] \\ -1 & , \text{if } x \in E_2 = E \cap [\frac{2}{3}, 1] \end{cases}$$

and for every $k \ge 1$ and $\alpha \in \mathcal{S}_k^2$:

$$\gamma_\alpha(x) = \begin{cases} 2^{k/2} & , \text{if } x \in E_{\alpha 1} \\ -2^{k/2} & , \text{if } x \in E_{\alpha 2} \\ 0 & , \text{if } x \notin E_\alpha \end{cases}$$

3.3. Theorem

The system Φ is orthonormal in the Hilbert space $L^2(E, H^s)$.

Proof: The normallity of the system Φ is easy to check. To see the orthogonallity it is enough to notice that, given two functions of the same or different generation, their supports are either disjoint or one of the functions is constant in the support of the other, so it would be enough to show that:

$$\int_E \gamma_0^h(x)\, dH^s(x) = \int_E \gamma_\alpha^h(x)\, dH^s(x) = 0$$

for every $k \geq 1$, $\alpha \in \mathcal{S}_k^\ell$ and $1 \leq h \leq \ell - 1$, which is trivial. □

Given a function $f \in L^1(E, H^s)$, we define its Fourier series with respect to the system Φ as:

$$f(x) \sim a_{-1}\gamma_{-1}(x) + \sum_{h=1}^{\ell-1} a_0^h \gamma_0^h(x) + \sum_{k=1}^{\infty} \sum_{\alpha\in\mathcal{S}_k^\ell} \sum_{h=1}^{\ell-1} a_\alpha^h \gamma_\alpha^h(x) \tag{2}$$

where:

$$\begin{aligned} a_{-1} &= \int_E f(y)\gamma_{-1}(y)\, dH^s(y) \quad ; \quad a_0^h = \int_E f(y)\gamma_0^h(y)\, dH^s(y) \ , \ 1 \leq h \leq \ell-1 \\ a_\alpha^h &= \int_E f(y)\gamma_\alpha^h(y)\, dH^s(y) \ , \ k \geq 1 \ , \ \alpha \in \mathcal{S}_k^\ell \ , \ 1 \leq h \leq \ell - 1 \end{aligned} \tag{3}$$

are the Fourier coefficients of f with respect to Φ, which we denote simply by:

$$f \sim \{a_{-1}, a_0^h, a_\alpha^h\}$$

We are now going to study the pointwise convergence of this series by considering the partial sums, which for every $m \geq 1$, $\beta \in \mathcal{S}_{m+1}^\ell$, $1 \leq p \leq \ell - 1$, are defined by:

$$\begin{aligned} S_{m+1}^{\beta;p} f(x) &= a_{-1}\gamma_{-1}(x) + \sum_{h=1}^{\ell-1} a_0^h\gamma_0^h(x) + \sum_{k=1}^{m} \sum_{\alpha\in\mathcal{S}_k^\ell} \sum_{h=1}^{\ell-1} a_\alpha^h\gamma_\alpha^h(x) + \\ &+ \sum_{\substack{\alpha\in\mathcal{S}_{m+1}^\ell \\ \alpha<\beta}} \sum_{h=1}^{\ell-1} a_\alpha^h\gamma_\alpha^h(x) + \sum_{h=1}^{p} a_\beta^h\gamma_\beta^h(x) \end{aligned} \tag{4}$$

Replacing (3) in the expression (4) and using the orthonormallity of the system Φ one finds that:

$$S_{m+1}^{\beta;p} f(x) = \int_E f(y) K_{m+1}^{\beta;p}(x,y)\, dH^s(y) \tag{5}$$

where:

$$\begin{aligned} K_{m+1}^{\beta;p}(x,y) &= \gamma_{-1}(x)\gamma_{-1}(y) + \sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) + \sum_{k=1}^{m} \sum_{\alpha\in\mathcal{S}_k^\ell} \sum_{h=1}^{\ell-1} \gamma_\alpha^h(x)\gamma_\alpha^h(y) + \\ &+ \sum_{\substack{\alpha\in\mathcal{S}_{m+1}^\ell \\ \alpha<\beta}} \sum_{h=1}^{\ell-1} \gamma_\alpha^h(x)\gamma_\alpha^h(y) + \sum_{h=1}^{p} \gamma_\beta^h(x)\gamma_\beta^h(y) \end{aligned} \tag{6}$$

is the Dirichlet kernel associated to the system Φ.

We shall give now some notations and definitions, as well as a technical Lemma, which will be useful to simplify later computations.

3.4. Notations and Definitions

For $1 \le q < p \le \ell - 1$, we define:

$$\begin{aligned} H(q,p) &= -C_q^{-1}(\sum_{i=1}^{q} r_i^s) r_{q+1}^{-s} + \sum_{h=q+1}^{p} C_h^{-1} \\ G(q,p) &= C_q^{-1}(\sum_{i=1}^{q} r_i^s)^2 r_{q+1}^{-2s} + \sum_{h=q+1}^{p} C_h^{-1} \end{aligned} \tag{7}$$

In particular, if $q = 0$ and $1 \le p \le \ell - 1$:

$$H(0,p) = G(0,p) = \sum_{h=1}^{p} C_h^{-1} \tag{8}$$

and if $q = p$, $1 \le p \le \ell - 1$:

$$\begin{aligned} H(p,p) &= -C_p^{-1}(\sum_{i=1}^{p} r_i^s) r_{p+1}^{-s} \\ &. \\ G(p,p) &= C_p^{-1}(\sum_{i=1}^{p} r_i^s)^2 r_{p+1}^{-2s} \end{aligned} \tag{9}$$

It is obvious that in the case $\ell = 2$ the definition (7) makes no sense.

3.5. Lemma

(a) $H(q,p) = -(\sum_{i=1}^{p+1} r_i^s)^{-1}$, $1 \le q \le p \le \ell - 1$.
(b) $G(q,p) = r_{q+1}^{-s} - (\sum_{i=1}^{p+1} r_i^s)^{-1}$, $1 \le q \le p \le \ell - 1$.
(c) $H(0,p) = G(0,p) = r_1^{-s} - (\sum_{i=1}^{p+1} r_i^s)^{-1}$, $1 \le p \le \ell - 1$.

Proof:
(a) If $q < p$ (it can only occur when $\ell > 2$), using (1) we have:

$$\begin{aligned} H(q,p) &= -\left(r_{q+1}^{-s} \sum_{i=1}^{q} r_i^s \sum_{i=1}^{q+1} r_i^s \right)^{-1} \left(\sum_{i=1}^{q} r_i^s \right) r_{q+1}^{-s} + \left(r_{q+2}^{-s} \sum_{i=1}^{q+1} r_i^s \sum_{i=1}^{q+2} r_i^s \right)^{-1} + \sum_{h=q+2}^{p} C_h^{-1} = \\ &= \frac{-1}{\sum_{i=1}^{q+1} r_i^s} + \frac{1}{r_{q+2}^{-s} \sum_{i=1}^{q+1} r_i^s \sum_{i=1}^{q+2} r_i^s} + \sum_{h=q+2}^{p} C_h^{-1} = \frac{-r_{q+2}^{-s} \sum_{i=1}^{q+2} r_i^s + 1}{C_{q+1}} + \sum_{h=q+2}^{p} C_h^{-1} = \\ &= -C_{q+1}^{-1} \left(\sum_{i=1}^{q+1} r_i^s \right) r_{q+2}^{-s} + \sum_{h=q+2}^{p} C_h^{-1} = H(q+1,p) \end{aligned}$$

and iterating the preceding process one gets:

$$H(q,p) = H(p,p)$$

and then:

$$H(p,p) = -C_p^{-1}\left(\sum_{i=1}^{p} r_i^s\right) r_{p+1}^{-s} = -\left(\sum_{i=1}^{p+1} r_i^s\right)^{-1}$$

(b) If $1 \le q \le p \le \ell - 1$:

$$\begin{aligned} G(q,p) &= C_q^{-1}\left(\sum_{i=1}^{q} r_i^s\right)^2 r_{q+1}^{-2s} + C_q^{-1}\left(\sum_{i=1}^{q} r_i^s\right) r_{q+1}^{-s} + H(q,p) = \\ &= C_q^{-1} r_{q+1}^{-s} C_q + H(q,p) = r_{q+1}^{-s} - \left(\sum_{i=1}^{p+1} r_i^s\right)^{-1} \end{aligned}$$

(c) If $1 \le p \le \ell - 1$:

$$\begin{aligned} H(0,p) = G(0,p) &= \sum_{h=1}^{p} C_h^{-1} = C_1^{-1} + C_1^{-1} r_1^s r_2^{-s} + H(1,p) = \\ &= \frac{1 + r_1^s r_2^{-s}}{C_1} + H(1,p) = r_1^{-s} - \left(\sum_{i=1}^{p+1} r_i^s\right)^{-1} \end{aligned}$$

□

We are going to compute, in the following Lemma, the Dirichlet kernel associated to the system Φ.

3.6. Lemma

(a)

$$\sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) = \begin{cases} r_q^{-s} - 1 & \text{, if } x,y \in E_q,\ 1 \le q \le \ell \\ -1 & \text{, if } x \in E_q,\ y \in E_j,\ q \ne j,\ 1 \le q,j \le \ell \end{cases}$$

(b) For every $k \ge 1$ and $\alpha \in \mathcal{S}_k^\ell$:

$$\sum_{h=1}^{\ell-1} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = \begin{cases} r_\alpha^{-s}(r_q^{-s} - 1) & \text{, if } x,y \in E_{\alpha q},\ 1 \le q \le \ell \\ -r_\alpha^{-s} & \text{, if } x \in E_{\alpha q},\ y \in E_{\alpha j},\ q \ne j,\ 1 \le q,j \le \ell \\ 0 & \text{, otherwise} \end{cases}$$

(c) For every $k \ge 1$, $\alpha \in \mathcal{S}_k^\ell$ and $1 \le p \le \ell - 1$:

$$\sum_{h=1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = \begin{cases} r_\alpha^{-s}\left[r_q^{-s} - \left(\sum_{i=1}^{p+1} r_i^s\right)^{-1}\right] & \text{, if } x,y \in E_{\alpha q},\ 1 \le q \le p+1 \\ -r_\alpha^{-s}\left(\sum_{i=1}^{p+1} r_i^s\right)^{-1} & \text{, if } x \in E_{\alpha q},\ y \in E_{\alpha j},\ q \ne j,\ 1 \le q,j \le p+1 \\ 0 & \text{, otherwise} \end{cases}$$

(d) For every $m \ge 1$:

$$K_m^{\ell,\dots,\ell;\ell-1}(x,y) = \begin{cases} r_\alpha^{-s} & \text{, if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}_{m+1}^\ell \\ 0 & \text{, otherwise} \end{cases}$$

(e) For every $m \geq 1$ and $\beta \in \mathcal{S}^{\ell}_{m+1}$:

$$K^{\beta;\ell-1}_{m+1}(x,y) = \begin{cases} r_\alpha^{-s} r_q^{-s} & \text{, if } x,y \in E_{\alpha q},\ \alpha \in \mathcal{S}^{\ell}_{m+1},\ \alpha \leq \beta,\ 1 \leq q \leq \ell \\ r_\alpha^{-s} & \text{, if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}^{\ell}_{m+1},\ \alpha > \beta \\ 0 & \text{, otherwise} \end{cases}$$

(f) For every $m \geq 1$, $\beta \in \mathcal{S}^{\ell}_{m+1}$ and $1 \leq p \leq \ell - 1$:

$$K^{\beta;p}_{m+1}(x,y) = \begin{cases} r_\alpha^{-s} r_q^{-s} & \text{, if } x,y \in E_{\alpha q},\ \alpha \in \mathcal{S}^{\ell}_{m+1},\ \alpha < \beta,\ 1 \leq q \leq \ell \\ r_\beta^{-s}\left[1 + r_q^{-s} - \left(\sum_{i=1}^{p+1} r_i^s\right)^{-1}\right] & \text{, if } x,y \in E_{\beta q},\ 1 \leq q \leq p+1 \\ r_\beta^{-s}\left[1 - \left(\sum_{i=1}^{p+1} r_i^s\right)^{-1}\right] & \text{, if } x \in E_{\beta q},\ y \in E_{\beta j},\ q \neq j,\ 1 \leq q,j \leq p+1 \\ r_\beta^{-s} & \text{, if } x \in E_{\beta q},\ y \in E_{\beta j},\ q > p+1 \text{ or } j > p+1 \\ r_\alpha^{-s} & \text{, if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}^{\ell}_{m+1},\ \alpha > \beta \\ 0 & \text{, otherwise} \end{cases}$$

Proof: First we notice that when $\ell = 2$ then (c) coincides with (b) and (f) with (e), and so in the proof of (c) and (f) we can assume that $\ell > 2$.
(a) Let $x \in E_q$, $y \in E_j$ and $j < q$. Then, by Definition 3.1 of the system Φ, we have:
(i) If $q = \ell$:

$$\begin{aligned} \sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) &= \gamma_0^{\ell-1}(x)\gamma_0^{\ell-1}(y) = -C_{\ell-1}^{-\frac{1}{2}}\left(\sum_{i=1}^{\ell-1} r_i^s\right) r_\ell^{-s} C_{\ell-1}^{-\frac{1}{2}} = \\ &= H(\ell-1,\ell-1) = -\left(\sum_{i=1}^{\ell} r_i^s\right)^{-1} = -1 \end{aligned}$$

using the definition of H and its evaluation in Lemma 3.5.
(ii) If $q < \ell$ (It can only happen if $\ell > 2$):

$$\begin{aligned} \sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) &= \sum_{h=q-1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) = -C_{q-1}^{-\frac{1}{2}}\left(\sum_{i=1}^{q-1} r_i^s\right) r_q^{-s} C_{q-1}^{-\frac{1}{2}} + \\ &+ \sum_{h=q}^{\ell-1} C_h^{-\frac{1}{2}} C_h^{-\frac{1}{2}} = H(q-1,\ell-1) = -1 \end{aligned}$$

By the symmetry of the kernel one gets the result for $q \neq j$.
Now let $x, y \in E_q$. We are going to distinguish three cases:
(i) If $q = \ell$:

$$\begin{aligned} \sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) &= \gamma_0^{\ell-1}(x)\gamma_0^{\ell-1}(y) = \left[-C_{\ell-1}^{-\frac{1}{2}}\left(\sum_{i=1}^{\ell-1} r_i^s\right) r_\ell^{-s}\right]^2 = \\ &= G(\ell-1,\ell-1) = r_\ell^{-s} - \left(\sum_{i=1}^{\ell} r_i^s\right)^{-1} = r_\ell^{-s} - 1 \end{aligned}$$

(ii) If $1 < q < \ell$ (if $\ell > 2$):

$$\sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) = \sum_{h=q-1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) = C_{q-1}^{-1}\left(\sum_{i=1}^{q-1} r_i^s\right)^2 r_q^{-2s} + \\ + \sum_{h=q}^{\ell-1} C_h^{-1} = G(q-1,\ell-1) = r_q^{-s} - 1$$

(iii) If $q = 1$:

$$\sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) = \sum_{h=1}^{\ell-1} C_h^{-1} = G(0,\ell-1) = r_1^{-s} - 1$$

With this (a) is proved.
(b) It can be obtained in the same way and following the same steps of (a).
(c) As we said at the begining of the proof, we shall assume $\ell > 2$. From the definition of Φ is clear that:

$$\bigcup_{h=1}^{p} support(\gamma_\alpha^h \times \gamma_\alpha^h) = \left(\bigcup_{i=1}^{p+1} E_{\alpha i}\right) \times \left(\bigcup_{i=1}^{p+1} E_{\alpha i}\right)$$

Suppose first that $x \in E_{\alpha q}$, $y \in E_{\alpha j}$ and $1 \le j < q \le p+1$. We are going to distinguish two cases:
(i) If $q = p+1$:

$$\sum_{h=1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = \gamma_\alpha^p(x)\gamma_\alpha^p(y) = r_\alpha^{-s} H(p,p)$$

(ii) If $q < p+1$:

$$\sum_{h=1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = \sum_{h=q-1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = r_\alpha^{-s} H(q-1,p)$$

Now let $x, y \in E_{\alpha q}$, $1 \le q \le p+1$, and we shall distinguish three cases:
(i) If $q = p+1$:

$$\sum_{h=1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = \gamma_\alpha^p(x)\gamma_\alpha^p(y) = r_\alpha^{-s} G(p,p)$$

(ii) If $1 < q < p+1$:

$$\sum_{h=1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = \sum_{h=q-1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = r_\alpha^{-s} G(q-1,p)$$

(iii) If $q = 1$:

$$\sum_{h=1}^{p} \gamma_\alpha^h(x)\gamma_\alpha^h(y) = r_\alpha^{-s} G(0,p)$$

Finally, using the symmetry of the kernel and Lemma 3.5 one immediately obtains the desired result.
(d) We shall prove it by induction on m. Using (a) it is clear that:

$$\gamma_{-1}(x)\gamma_{-1}(y) + \sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) = \begin{cases} r_i^{-s} & \text{, if } x, y \in E_i,\ 1 \le i \le \ell \\ 0 & \text{, otherwise} \end{cases} \tag{10}$$

For every i, $1 \leq i \leq \ell$, $\sum_{h=1}^{\ell-1} \gamma_i^h(x)\gamma_i^h(y)$ has support in $E_i \times E_i$. Then, using (b) and (10), we get:

$$\begin{aligned} K_1^{\ell;\ell-1}(x,y) &= \gamma_{-1}(x)\gamma_{-1}(y) + \sum_{h=1}^{\ell-1} \gamma_0^h(x)\gamma_0^h(y) + \sum_{i=1}^{\ell}\sum_{h=1}^{\ell-1} \gamma_i^h(x)\gamma_i^h(y) = \\ &= \begin{cases} r_i^{-s} + r_i^{-s}(r_j^{-s} - 1) & , \text{if } x,y \in E_{ij},\ 1 \leq i,j \leq \ell \\ r_i^{-s} - r_i^{-s} & , \text{if } x \in E_{iq},\ y \in E_{ij},\ 1 \leq i,j,q \leq \ell,\ j \neq q \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

That is:

$$K_1^{\ell;\ell-1}(x,y) = \begin{cases} r_\alpha^{-s} & , \text{if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}_2^\ell \\ 0 & , \text{otherwise} \end{cases}$$

and then (d) is true for $m = 1$. Suppose it is true for $m-1$, that is:

$$K_{m-1}^{\ell,\dots,\ell;\ell-1}(x,y) = \begin{cases} r_\alpha^{-s} & , \text{if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}_m^\ell \\ 0 & , \text{otherwise} \end{cases}$$

Then:

$$K_m^{\ell,\dots,\ell;\ell-1}(x,y) = K_{m-1}^{\ell,\dots,\ell;\ell-1}(x,y) + \sum_{\alpha \in \mathcal{S}_m^\ell} \sum_{h=1}^{\ell-1} \gamma_\alpha^h(x)\gamma_\alpha^h(y)$$

and since $\bigcup_{h=1}^{\ell-1} support(\gamma_\alpha^h \times \gamma_\alpha^h) = E_\alpha \times E_\alpha$, by the induction hypothesis and (b) we have:

$$K_m^{\ell,\dots,\ell;\ell-1}(x,y) = \begin{cases} r_\alpha^{-s} + r_\alpha^{-s}(r_i^{-s} - 1) & , \text{if } x,y \in E_{\alpha i},\ \alpha \in \mathcal{S}_m^\ell,\ 1 \leq i \leq \ell \\ r_\alpha^{-s} - r_\alpha^{-s} & , \text{if } x \in E_{\alpha q},\ y \in E_{\alpha j},\ \alpha \in \mathcal{S}_m^\ell,\ 1 \leq q,j \leq \ell,\ q \neq j \\ 0 & , \text{otherwise} \end{cases}$$

from which one immediately obtains the result.
(e) We have that:

$$K_{m+1}^{\beta;\ell-1}(x,y) = K_m^{\ell,\dots,\ell;\ell-1}(x,y) + \sum_{\substack{\alpha \in \mathcal{S}_{m+1}^\ell \\ \alpha \leq \beta}} \sum_{h=1}^{\ell-1} \gamma_\alpha^h(x)\gamma_\alpha^h(y)$$

and one can notice easily that:

$$K_{m+1}^{\beta;\ell-1}(x,y) = \begin{cases} K_{m+1}^{\ell,\dots,\ell;\ell-1}(x,y) & , \text{if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}_{m+1}^\ell,\ \alpha \leq \beta \\ K_m^{\ell,\dots,\ell;\ell-1}(x,y) & , \text{if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}_{m+1}^\ell,\ \alpha > \beta \\ 0 & , \text{otherwise} \end{cases}$$

Then, using (d), one obtains the desired result.
(f) We may suppose, as we said before, that $\ell > 2$. The proof of this result is an immediate consequence of (c), (e) and the fact that:

$$K_{m+1}^{\beta;p}(x,y) = \begin{cases} K_{m+1}^{\beta;\ell-1}(x,y) & , \text{if } x,y \in E_\alpha,\ \alpha \in \mathcal{S}_{m+1}^\ell,\ \alpha \neq \beta \\ K_{m+1}^{\beta^-;\ell-1}(x,y) + \sum_{h=1}^{p} \gamma_\beta^h(x)\gamma_\beta^h(y) & , \text{if } x,y \in E_\beta \\ 0 & , \text{otherwise} \end{cases}$$

□

We are going to compute, in the following Lemma, the partial sums of the Fourier series of a function $f \in L^1(E, H^s)$.

3.7. Lemma

For every $m \geq 1$, $\beta \in \mathcal{S}_{m+1}^{\ell}$ and $1 \leq p \leq \ell - 1$, we have:

$$
S_{m+1}^{\beta;p} f(x) = \begin{cases}
\frac{1}{H^s(E_{\alpha q})} \int_{E_{\alpha q}} f(y)\, dH^s(y) & \text{, if } x \in E_{\alpha q},\ \alpha \in \mathcal{S}_{m+1}^{\ell},\ \alpha < \beta,\ 1 \leq q \leq \ell \\
\frac{1}{H^s(E_\beta)} \int_{E_\beta} f(y)\, dH^s(y) + & \\
\quad + \frac{1}{H^s(E_{\beta q})} \int_{E_{\beta q}} f(y)\, dH^s(y) - & \\
\quad - \frac{1}{H^s(\bigcup_{i=1}^{p+1} E_{\beta i})} \int_{\bigcup_{i=1}^{p+1} E_{\beta i}} f(y)\, dH^s(y) & \text{, if } x \in E_{\beta q},\ 1 \leq q \leq p+1 \\
\frac{1}{H^s(E_\beta)} \int_{E_\beta} f(y)\, dH^s(y) & \text{, if } x \in E_{\beta q},\ p+1 < q \leq \ell \\
\frac{1}{H^s(E_\alpha)} \int_{E_\alpha} f(y)\, dH^s(y) & \text{, if } x \in E_\alpha,\ \alpha \in \mathcal{S}_{m+1}^{\ell},\ \alpha > \beta
\end{cases}
$$

Proof: Since:

$$
S_{m+1}^{\beta;p} f(x) = \int_E f(y) K_{m+1}^{\beta;p}(x,y)\, dH^s(y)
$$

and considering the result (f) of Lemma 3.6, we have:
If $x \in E_{\alpha q}$, $\alpha \in \mathcal{S}_{m+1}^{\ell}$, $\alpha < \beta$ and $1 \leq q \leq \ell$:

$$
S_{m+1}^{\beta;p} f(x) = \int_E f(y) r_\alpha^{-s} r_q^{-s} \chi_{E_{\alpha q}}(y)\, dH^s(y) = \frac{1}{H^s(E_{\alpha q})} \int_{E_{\alpha q}} f(y)\, dH^s(y)
$$

because if $H^s(E) = 1$, then $H^s(E_\alpha) = r_\alpha^s$.
If $x \in E_{\alpha q}$, $1 \leq q \leq p+1$:

$$
\begin{aligned}
S_{m+1}^{\beta;p} f(x) &= \int_E f(y) \Bigg\{ \sum_{\substack{j=1 \\ j \neq q}}^{p+1} r_\beta^{-s} \left[1 - \left(\sum_{i=1}^{p+1} r_i^s \right)^{-1} \right] \chi_{E_{\beta j}}(y) + \\
&+ r_\beta^{-s} \left[1 + r_q^{-s} - \left(\sum_{i=1}^{p+1} r_i^s \right)^{-1} \right] \chi_{E_{\beta q}}(y) + \left(\sum_{j=p+2}^{\ell} r_\beta^{-s} \right) \chi_{E_{\beta j}}(y) \Bigg\}\, dH^s(y) = \\
&= \frac{1}{r_\beta^s} \int_E f(y) \left(\sum_{\substack{j=1 \\ j \neq q}}^{p+1} \chi_{E_{\beta j}}(y) + \chi_{E_{\beta q}}(y) + \sum_{j=p+2}^{\ell} \chi_{E_{\beta j}}(y) \right) dH^s(y) + \\
&+ \frac{1}{r_\beta^s r_q^s} \int_E f(y) \chi_{E_{\beta q}}(y)\, dH^s(y) - \\
&- \frac{1}{r_\beta^s \sum_{i=1}^{p+1} r_i^s} \int_E f(y) \left(\sum_{\substack{j=1 \\ j \neq q}}^{p+1} \chi_{E_{\beta j}}(y) + \chi_{E_{\beta q}}(y) \right) dH^s(y)
\end{aligned}
$$

and we obtain the announced expression. The other two cases are similar to the first one. □

We can state now the following theorem which can be obtained immediately by applying the Differentiation Theorem 2.4 to the expression of the partial sums obtained in Lemma 3.7.

3.8. Theorem

For every function $f \in L^1(E, H^s)$, the partial sums of its Fourier series, with respect to Φ, converges to f at H^s-almost every point of E.

3.9. Corollary

The system Φ is L^1-complete.

Proof: Suppose that $f \in L^1(E, H^s)$ is orthogonal to every function of the system Φ. Then, it is clear that:

$$S_{m+1}^{\beta;p} f(x) = 0 \quad , \text{ for all } x \in E$$

for every $m \geq 1$, $\beta \in \mathcal{S}_{m+1}^{\ell}$ and $1 \leq p \leq \ell - 1$. Then, using Theorem 3.8, we have that $f(x) = 0$ in H^s-almost every $x \in E$. □

3.10. Remark: Since $H^s(E) < \infty$, we have that $L^p(E, H^s) \subset L^1(E, H^s)$ for every p, $1 < p \leq \infty$. Then, Corollary 3.9 assures us that the system Φ is L^p-complete, $1 \leq p \leq \infty$, and in particular that Φ is L^2-complete or that it is an orthonormal maximal system in $L^2(E, H^s)$.

We shall see now that the Fourier coefficients of every function of $L^1(E, H^s)$ converge, in some sense, to zero. Since the intersection of two different subsets of E of the same generation is empty, then the set E may be identified with $\mathcal{S}_\infty^\ell$ as follows. Given $x \in E$ there is a unique infinite sequence $\alpha(x) \in \mathcal{S}_\infty^\ell$ such that:

$$x \in \bigcap_{m=1}^{\infty} E_{\alpha(x)[m]}$$

We construct the sequence $\alpha(x)$ with the subindices of the sets of each generation which contain x. Conversely, given $\alpha \in \mathcal{S}_\infty^\ell$, we assign to α the unique point $x \in E$ such that:

$$x \in \bigcap_{m=1}^{\infty} E_{\alpha[m]}$$

Without the additional hypothesis that different sets of the same generation have empty intersection the identification may be done in the same way, but now there will be a set of points of null H^s-measure and each one of them will be assigned with more than one sequence (such points may be excluded).

3.11. Theorem

Let $f \in L^1(E, H^s)$ and $\{a_{-1}, a_0^h, a_\alpha^h\}$ their Fourier coefficients with respect to Φ. Then, for H^s-a.e. $x \in E$:

$$\lim_{m \to \infty} a_{\alpha(x)[m]}^h = 0$$

for every h, $1 \leq h \leq \ell - 1$.

Proof: For every $k \geq 1$, $\alpha \in \mathcal{S}_k^\ell$ and $1 \leq h \leq \ell - 1$, we have:

$$\begin{aligned} a_\alpha^h &= \int_E f(y) \gamma_\alpha^h(y) \, dH^s(y) = \\ &= C_h^{-\frac{1}{2}} r_\alpha^{-\frac{s}{2}} \left(\sum_{j=1}^{h} \int_{E_{\alpha j}} f(y) \, dH^s(y) - r_{h+1}^{-s} \sum_{i=1}^{h} r_i^s \int_{E_{\alpha, h+1}} f(y) \, dH^s(y) \right) = \end{aligned}$$

$$= C_h^{-\frac{1}{2}} r_\alpha^{-\frac{s}{2}} \left(\int_{\bigcup_{j=1}^h E_{\alpha j}} f(y)\, dH^s(y) - r_{h+1}^{-s} \sum_{i=1}^h r_i^s \int_{E_{\alpha,h+1}} f(y)\, dH^s(y) \right)$$

If $K = \max\{1\ ,\ \max_{1 \le h \le \ell-1}\{r_{h+1}^{-s} \sum_{i=1}^h r_i^s\}\}$, then:

$$|a_\alpha^h| \le C_h^{-\frac{1}{2}} r_\alpha^{-\frac{s}{2}} K \int_{E_\alpha} |f(y)|\, dH^s(y) = C_h^{-\frac{1}{2}} r_\alpha^{\frac{s}{2}} K \frac{1}{H^s(E_\alpha)} \int_{E_\alpha} |f(y)|\, dH^s(y)$$

Since $f \in L^1$, we have that $|f| \in L^1$ and applying the Differentiation Theorem 2.4 we get that at H^s-almost every $x \in E$:

$$\lim_{m\to\infty} \frac{1}{H^s(E_{\alpha(x)[m]})} \int_{E_{\alpha(x)[m]}} |f(y)|\, dH^s(y) = |f(x)|$$

and since:

$$\lim_{m\to\infty} r_{\alpha(x)[m]}^{\frac{s}{2}} = 0$$

we obtain the result of the theorem. □

We shall now study, in the next theorems, the convergence in L^p-norm, $1 \le p \le \infty$, of the Fourier series of the functions $f \in L^p(E, H^s)$.

Since Φ is a L^2-complete system, we can apply the classic results of the ordinary theory of Hilbert spaces (Bessel inequality, Riesz-Fischer theorem,...) and state, then, the following theorem:

3.12. Theorem

If $f \in L^2(E, H^s)$ and $\{a_{-1}, a_0^h, a_\alpha^h\}$ are their Fourier coefficients with respect to Φ, we have:

(a) $||f||_2^2 = (a_{-1})^2 + \sum_{h=1}^{\ell-1}(a_\alpha^h)^2 + \sum_{k=1}^\infty \sum_{\alpha\in S_k^\ell} \sum_{h=1}^{\ell-1}(a_\alpha^h)^2 < \infty$.

(b) $||S_{m+1}^{\beta;p} f - f||_2 \longrightarrow 0$.

Moreover, if $F \in L^2(E, H^s)$ and $\{A_{-1}, A_0^h, A_\alpha^h\}$ are their Fourier coefficients with respect to Φ, then:

$$\int_E f(y)F(y)\, dH^s(y) = a_{-1}A_{-1} + \sum_{h=1}^{\ell-1} a_0^h A_0^h + \sum_{k=1}^\infty \sum_{\alpha\in S_k^\ell} \sum_{h=1}^{\ell-1} a_\alpha^h A_\alpha^h$$

where the last series is absolutely convergent.

And if $\{c_{-1}, c_0^h, c_\alpha^h\}$ is a sequence of real numbers which satisfies:

$$(c_{-1})^2 + \sum_{h=1}^{\ell-1}(c_0^h)^2 + \sum_{k=1}^\infty \sum_{\alpha\in S_k^\ell} \sum_{h=1}^{\ell-1}(c_\alpha^h)^2 < \infty$$

then there is a unique function $f \in L^2(E, H^s)$ such that $\{c_{-1}, c_0^h, c_\alpha^h\}$ are their Fourier coefficients, and then f satisfies (a) and (b).

3.13. Theorem

Let p, $1 \le p \le \infty$. If $\{c_{-1}, c_0^h, c_\alpha^h\}$ is a sequence of real numbers which satisfies:

$$|c_{-1}| + \sum_{h=1}^{\ell-1} |c_0^h| \cdot ||\gamma_0^h||_p + \sum_{k=1}^\infty \sum_{\alpha\in S_k^\ell} \sum_{h=1}^{\ell-1} |c_\alpha^h| \cdot ||\gamma_\alpha^h||_p < \infty \tag{11}$$

then there is a unique function $f \in L^p(E, H^s)$ such that $\{c_{-1}, c_0^h, c_\alpha^h\}$ are their Fourier coefficients, and we have:

$$||S_{m+1}^{\beta;p} f - f||_p \longrightarrow 0$$

Moreover, if we have a function $f \in L^p$ and their Fourier coefficients $\{a_{-1}, a_0^h, a_\alpha^h\}$ verify (11), then the Fourier series of f converges to f in L^p-norm.

Proof: In order to simplify the notation we shall suppose that $\Phi = \{\gamma_k\}_{k\geq 1}$. Let $\{c_k\}_{k\geq 1}$ a sequence of real numbers verifying (11), that is:

$$\sum_{k=1}^{\infty} |c_k| \cdot ||\gamma_k||_p < \infty \tag{12}$$

Since $\Phi \subset L^p$, by (12), it is immediate to check that if we denote:

$$f_N(x) = \sum_{k=1}^{N} c_k \gamma_k(x)$$

the sequence $\{f_N\}_{N\geq 1}$ is a Cauchy sequence in the Banach space $L^p(E, H^s)$. Then there will be a function $f \in L^p$ such that:

$$\lim_{N\to\infty} ||f_N - f||_p = 0 \tag{13}$$

To see that $\{c_k\}_{k\geq 1}$ are the Fourier coefficients of f it is enough to consider (13), and the facts that $\Phi \subset L^{p'}$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ and that $\gamma_k \longrightarrow_{N\to\infty} \gamma_k$ in $L^{p'}$-norm, since in this case (see [4,pp.9]), for every $k \geq 1$, we have that:

$$\int_E f(y)\gamma_k(y)\, dH^s(y) = \lim_{N\to\infty} \int_E f_N(y)\gamma_k(y)\, dH^s(y) = c_k$$

The uniqueness of the function and the rest of the proof may be deduced from the L^p-completeness of Φ. □

4. Examples

We shall now see some examples in which we compute the Fourier coefficients of certain functions, and we build functions from their Fourier coefficients.

In the rest of the paper $E \subset [0,1] \subset \mathbb{R}$ will be the classic Cantor set and:

$$\Phi = \{\gamma_{-1}, \gamma_0\} \cup \{\gamma_\alpha \ : \ \alpha \in \mathcal{S}_k^2 \ , \ k \geq 1\}$$

the system associated to it (see 3.2(d)).

Example 1: If $f : E \longrightarrow \mathbb{R}$, $f(x) = x$, then we are going to see that:

$$f(x) = \frac{1}{2}\gamma_{-1}(x) - \frac{1}{3}\gamma_0(x) - \frac{1}{3}\sum_{k=1}^{\infty} \frac{1}{(3\sqrt{2})^k} \sum_{\alpha\in\mathcal{S}_k^2} \gamma_\alpha(x)$$

with convergence at H^s-almost every point $x \in E$.

Proof: The computation of the coefficients is based on the two following facts, which are easy to check.
(i) Let $A \in \mathbb{R}$ be a constant and $g : E \longrightarrow \mathbb{R}$. Then, for every $k \geq 1$ and $\alpha \in \mathcal{S}_k^2$:

$$\int_{E_\alpha} [A + g(y)]\gamma_\alpha(y)\, dH^s(y) = \int_{E_\alpha} g(y)\gamma_\alpha(y)\, dH^s(y)$$

since $\int_{E_\alpha} \gamma_\alpha(y)\, dH^s(y) = 0$.
(ii) If $g : E \longrightarrow \mathbb{R}$ is such that $g(x) = A + mx$, being $A, m \in \mathbb{R}$ constants, then for every $k \geq 1$ and $\alpha \in \mathcal{S}_k^2$ we have:

$$\int_{E_\alpha} g(y)\, dH^s(y) = \frac{g(S_\alpha(0)) + g(S_\alpha(1))}{2} H^s(E_\alpha)$$

by the distribution of the H^s-measure on the set E_α and because of $E_\alpha = [S_\alpha(0), S_\alpha(1)] \cap E$.

We shall compute the coefficients:

$$\begin{aligned} a_{-1} &= \int_E y\, dH^s(y) = \frac{0+1}{2} H^s(E) = \frac{1}{2} \\ a_0 &= \int_E y\gamma_0(y)\, dH^s(y) = \int_{E_1} y\, dH^s(y) - \int_{E_2} y\, dH^s(y) = \frac{0+\frac{1}{3}}{2}\frac{1}{2} - \frac{\frac{2}{3}+1}{2}\frac{1}{2} = \frac{-1}{3} \end{aligned}$$

since $E_1 = [0, \frac{1}{3}] \cap E$ and $E_2 = [\frac{2}{3}, 1] \cap E$. And for every $k \geq 1$ and $\alpha \in \mathcal{S}_k^2$ we have:

$$\begin{aligned} a_\alpha &= a_{1,\ldots,1} = \int_{E_{1,\ldots,1}} y\gamma_{1,\ldots,1}(y)\, dH^s(y) = \\ &= 2^{\frac{k}{2}} \int_{E_{1,\ldots,1,1}} y\, dH^s(y) - 2^{\frac{k}{2}} \int_{E_{1,\ldots,1,2}} y\, dH^s(y) = \\ &= 2^{\frac{k}{2}} \left(\frac{0 + (\frac{1}{3})^{k+1}}{2} \frac{1}{2^{k+1}} - \frac{(\frac{1}{3})^k \frac{2}{3} + (\frac{1}{3})^k}{2} \frac{1}{2^{k+1}} \right) = \frac{-1}{3(3\sqrt{2})^k} \end{aligned}$$

□

Example 2: If we consider that $c_{-1} = c_0 = 1$ and $c_\alpha = 2^{-k}$ for every $\alpha \in \mathcal{S}_k^2$, $k \geq 1$, then:

$$(c_{-1})^2 + (c_0)^2 + \sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{S}_k^2} (c_\alpha)^2 = 2 + \sum_{k=1}^{\infty} 2^{-k} = 3 < +\infty$$

and by Theorem 3.12, there is a unique function $f \in L^2(E, H^s)$, $||f||_2 = \sqrt{3}$, whose Fourier coefficients are $\{c_{-1}, c_0, c_\alpha\}$, and moreover:

$$f(x) = \gamma_{-1}(x) + \gamma_0(x) + \sum_{k=1}^{\infty} 2^{-k} \sum_{\alpha \in \mathcal{S}_k^2} \gamma_\alpha(x)$$

with convergence in H^s-almost every point and in L^2-norm. Using the definition of the system Φ (see 3.2(d)) we have:

$$f(x) = 1 - \sum_{k=1}^{\infty} \frac{(-1)^{\alpha(x)_k}}{2^{\frac{k-1}{2}}}$$

where $\alpha(x) = (i_1, i_2, \ldots) \in \mathcal{S}_\infty^2$ is the unique infinite sequence which determines x and $\alpha(x)_k = i_k$, for all $k \geq 1$. □

Example 3: If we consider $c_{-1} = c_0 = 1$ and for every $k \geq 1$ and $\alpha \in \mathcal{S}_k^2$:

$$c_\alpha = \begin{cases} 1 & , \text{if } \alpha = (1, ..., 1) \in \mathcal{S}_k^2 \\ 0 & , \text{if } \alpha \in \mathcal{S}_k^2 \setminus \{(1, ..., 1)\} \end{cases}$$

then:

$$(c_{-1})^2 + (c_0)^2 + \sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{S}_k^2} (c_\alpha)^2 = 2 + \sum_{k=1}^{\infty} 1 = +\infty$$

But if we compute the L^p-norm, $1 \leq p < \infty$, of the functions of the system Φ (see 3.2(d)), we have that:

$$\begin{aligned} ||\gamma_{-1}||_p = ||\gamma_0||_p &= 1 \\ ||\gamma_\alpha||_p &= \left(\int_E |\gamma_\alpha|^p \, dH^s\right)^{\frac{1}{p}} = \left(2^{\frac{pk}{2}} \frac{1}{2^{k+1}} + 2^{\frac{pk}{2}} \frac{1}{2^{k+1}}\right)^{\frac{1}{p}} = 2^{\frac{p+2}{2p}k} \end{aligned}$$

and then:

$$|c_{-1}| \, ||\gamma_{-1}||_1 + |c_0| \, ||\gamma_0||_1 + \sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{S}_k^2} |c_\alpha| \, ||\gamma_\alpha||_1 = 2 + \sum_{k=1}^{\infty} 2^{\frac{-k}{2}} < \infty$$

Then, by the Theorems 3.12 and 3.13, there is a unique function $f \in L^1 \setminus L^2$ such that $\{c_{-1}, c_0, c_\alpha\}$ are their Fourier coefficients and:

$$f(x) = \gamma_{-1}(x) + \gamma_0(x) + \sum_{k=1}^{\infty} \gamma_{1,\ldots,1}(x)$$

with convergence in L^1-norm and at H^s-almost every point. It is easy to check that:

$$f(x) = \begin{cases} 0 & , \text{if } x \in E_2 \\ \sqrt{2}(2^{\frac{k}{2}} - 1) & , \text{if } x \in E_{\alpha 2}, \ \alpha = (1, ..., 1) \in \mathcal{S}_k^2, \ k \geq 1 \end{cases}$$

□

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