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THE INVERSION OF APPROXIMATE AND DYADIC DERIVATIVES USING AN EXTENSION OF THE HENSTOCK INTEGRAL

The Henstock integral integrates all ordinary derivatives and recovers the primitive. This fact follows quite easily from the definition. It is natural to ask whether or not it is possible to modify the Henstock integral so that it integrates other types of derivatives. This paper addresses this problem for approximate derivatives and dyadic derivatives. For similar approaches to these problems see Lee [3] and Pacquement [4]. We also prove a convergence theorem for the newly defined integrals.

We will assume that the reader is familiar with the Henstock integral. Throughout this paper $\mathcal{P}$ will denote a finite collection of non-overlapping tagged intervals in $[a,b]$. For $\mathcal{P} = \{(t_i,[c_i,d_i]) : 1 \le i \le N\}$, we will write

$$f(\mathcal{P}) = \sum_{i=1}^{N} f(t_i)(d_i - c_i), \quad F(\mathcal{P}) = \sum_{i=1}^{N}\big(F(d_i) - F(c_i)\big), \quad \text{and} \quad \mu(\mathcal{P}) = \sum_{i=1}^{N}(d_i - c_i).$$

This is an abuse of notation, but it is quite convenient. Given a set E and a point t, let $\rho(t,E)$ be the distance from t to E, CE be the complement of E, and $\overline{E}$ be the closure of E.

We begin by considering the approximate derivative. Recall that F has an approximate derivative at x if there exists a measurable set E having x as a point of density such that $\lim\limits_{\substack{t\to x\\ t\in E}} \dfrac{F(t)-F(x)}{t-x}$ exists. To define an integral that recovers approximate derivatives, we must limit the collection of tagged intervals. This is the content of the first definition.

DEFINITION 1: A distribution $\mathcal{S}$ on $[a,b]$ is a collection of measurable sets $\{S_x : x \in [a,b]\}$ in $[a,b]$ such that $x \in S_x$ and x is a point of density of S_x. For each $x \in [a,b]$, let $\mathcal{I}_x = \{[c,d] : x \in [c,d] \text{ and } c,d \in S_x\}$. Let δ be a positive function defined on $[a,b]$. A collection $\mathcal{P}$ of tagged intervals is $\mathcal{S}$–subordinate to δ if $d-c < \delta(x)$ and $[c,d] \in \mathcal{I}_x$ whenever $(x,[c,d]) \in \mathcal{P}$. If in addition $\mathcal{P}$ is a partition of $[a,b]$, then we will often write $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a,b]$.

Unless stated otherwise $\mathcal{S}$ will represent a fixed but arbitrary distribution on $[a,b]$. We must first prove that partitions $\mathcal{S}$–subordinate to δ exist for each positive function δ. We need the following lemma which is due to Romanovski [5]. The proof is essentially an application of the Heine-Borel theorem.

LEMMA 2: (Romanovski's Lemma) Let $\mathcal{F}$ be a family of open intervals in (a,b) and suppose that $\mathcal{F}$ has the following properties:

(1) If (α,β) and (β,γ) belong to $\mathcal{F}$, then (α,γ) belongs to $\mathcal{F}$.

(2) If (α,β) belongs to $\mathcal{F}$, then every open interval in (α,β) belongs to $\mathcal{F}$.

(3) If (α,β) belongs to $\mathcal{F}$ for every interval $[\alpha,\beta] \subset (c,d)$, then (c,d) belongs to $\mathcal{F}$.

(4) If all of the open intervals in (a,b) contiguous to the perfect set $E \subset [a,b]$ belong to $\mathcal{F}$, then there exists an interval I in $\mathcal{F}$ such that $I \cap E \neq \emptyset$.

Then $\mathcal{F}$ contains the interval (a,b).

PROOF: By applying condition (4) to the set $[a,b]$, we find that $\mathcal{F}$ is non-empty. Let $H = (a,b) - \bigcup_{I\in\mathcal{F}} I$ and note that $\overline{H} - H \subset \{a,b\}$. Let $(a,b) - H = \bigcup_k (c_k,d_k)$.

We first prove that each interval (c_k,d_k) belongs to $\mathcal{F}$. To this end, fix k and let $[\alpha,\beta] \subset (c_k,d_k)$. For each t in $[\alpha,\beta]$, there exists an interval I_t in $\mathcal{F}$ that contains t. The collection $\{I_t : t \in [\alpha,\beta]\}$ is an open cover of $[\alpha,\beta]$ and since $[\alpha,\beta]$ is compact, there exists a finite subcover $\{I_{t_i} : 1 \le i \le M\}$. Let $\{s_j : 0 \le j \le N\}$ be the set that contains $\{\alpha,\beta\}$ and all of the endpoints of the I_{t_i}'s that belong to $[\alpha,\beta]$, and assume that the points are in increasing order. By condition (2), each of the intervals (s_{j-1},s_j) for $1 \le j \le N$ belongs to $\mathcal{F}$. By repeated application of condition (1), we find that (α,β) belongs to $\mathcal{F}$. By condition (3), we conclude that (c_k,d_k) belongs to $\mathcal{F}$.

Now condition (1) implies that the set $\overline{H}$ is perfect. In addition, each of the intervals contiguous to $\overline{H}$ in (a,b) belongs to $\mathcal{F}$. By condition (4), the set H must be empty. Repeating the argument of the second paragraph, we find that (a,b) belongs to $\mathcal{F}$.

LEMMA 3: If $\mathcal{S}$ is a distribution on $[a,b]$, then for each positive function δ on $[a,b]$ there exists a tagged partition of $[a,b]$ that is $\mathcal{S}$–subordinate to δ.

PROOF: Let $\mathcal{F}$ be the collection of all open intervals (u,v) in (a,b) for which each interval $[s,t] \subset [u,v]$ has a tagged partition that is $\mathcal{S}$-subordinate to δ. We will verify that $\mathcal{F}$ satisfies the four conditions of Romanovski's Lemma. It is clear that $\mathcal{F}$ satisfies conditions (1) and (2). Suppose that $(\alpha,\beta) \in \mathcal{F}$ for each interval $[\alpha,\beta] \subset (c,d)$. Choose $c_1 \in (c,d) \cap S_c$ such that $c_1 - c < \delta(c)$ and $d_1 \in (c_1,d) \cap S_d$ such that $d - d_1 < \delta(d)$. Let $\mathcal{P}_1$ be a tagged partition of $[c_1,d_1]$ that is $\mathcal{S}$-subordinate to δ. Then $\mathcal{P} = (c,[c,c_1]) \cup \mathcal{P}_1 \cup (d,[d_1,d])$ is a tagged partition of $[c,d]$ that is $\mathcal{S}$-subordinate to δ. It follows easily that $(c,d) \in \mathcal{F}$. This shows that $\mathcal{F}$ satisfies condition (3).

Now suppose that E is a perfect set in $[a,b]$ and that each interval contiguous to E in (a,b) belongs to $\mathcal{F}$. Since x is a point of density of S_x, for each $x \in [a,b]$ there exists a positive number $\eta_x < \delta(x)$ such that $0 < h \le \eta_x$ implies

$$\mu(S_x \cap [x-h,x]) > h/2 \quad \text{and} \quad \mu(S_x \cap [x,x+h]) > h/2.$$

For each positive integer n, let $A_n = \{x \in E : \eta_x \ge 1/n\}$, and for each integer i, let $A_{ni} = A_n \cap [(i-1)/n, i/n]$. Then $E = \bigcup_n A_n$ and $E = \bigcup_n \bigcup_i \overline{A_{ni}}$. By the Baire Category Theorem, there exist an interval (u, v) with $u, v \in E$ and a set A_{mj} such that $E \cap (u, v) \neq \emptyset$ and $E \cap [u, v] = \overline{A_{mj}} \cap [u, v]$. Let $[c, d] \subset [u, v]$. We will consider one of several cases; the others are similar. Suppose that $c \in E$ and $d \notin E$. Choose an interval (s, t) contiguous to E in (u, v) such that $d \in (s, t)$. Since $c, s \in E$, there exists an integer $p \geq m$ such that $c, s \in A_p$. Since A_{mj} is dense in $E \cap [u, v]$, there exist $c_1 \in [c, c + 1/p] \cap A_{mj}$ and $s_1 \in [s - 1/p, s] \cap A_{mj}$ with $c_1 < s_1$. Now $[c, d] = [c, c_1] \cup [c_1, s_1] \cup [s_1, s] \cup [s, d]$, so it is sufficient to prove that each of these intervals has a tagged partition that is $\mathcal{S}$–subordinate to δ.

Since $[s, d] \subset [s, t]$ and $(s, t) \in \mathcal{F}$, the interval $[s, d]$ has a tagged partition that is $\mathcal{S}$–subordinate to δ. Since $c, c_1 \in A_p$ and $c_1 - c \leq 1/p$, we have

$$\mu(S_c \cap [c, c_1]) > (c_1 - c)/2 \quad \text{and} \quad \mu(S_{c_1} \cap [c, c_1]) > (c_1 - c)/2.$$

Choose a point $y \in S_c \cap (c, c_1) \cap S_{c_1}$ and let $\mathcal{P} = (c, [c, y]) \cup (c_1, [y, c_1])$. Then $\mathcal{P}$ is a tagged partition of $[c, c_1]$ that is $\mathcal{S}$–subordinate to δ. Similarly, the intervals $[c_1, s_1]$ and $[s_1, s]$ have partitions that are $\mathcal{S}$–subordinate to δ. Hence $[c, d]$ has a partition that is $\mathcal{S}$–subordinate to δ. We conclude that $(u, v) \in \mathcal{F}$ and this shows that $\mathcal{F}$ satisfies condition (4). This completes the proof.

DEFINITION 4: The function $f : [a, b] \to R$ is $\mathcal{S}$–Henstock integrable on $[a, b]$ if there exists a real number α such that for each $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $|f(\mathcal{P}) - \alpha| < \epsilon$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a, b]$.

It is clear that the number α is unique. If f is Henstock integrable on $[a, b]$, then f is $\mathcal{S}$–Henstock integrable on $[a, b]$ for any distribution $\mathcal{S}$ on $[a, b]$ and the integrals are equal. The function cf is $\mathcal{S}$–Henstock integrable on $[a, b]$ for any constant c provided that the function f is. For the sum of two functions, we first note that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two distributions on $[a, b]$, then $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 = \{S_x^1 \cap S_x^2 : x \in [a, b]\}$ is a distribution on $[a, b]$. This yields the following result.

THEOREM 5: Let f and g be functions mapping $[a, b]$ into R. If f is $\mathcal{S}_1$–Henstock integrable on $[a, b]$ and if g is $\mathcal{S}_2$–Henstock integrable on $[a, b]$, then $f + g$ is $\mathcal{S}$–Henstock integrable on $[a, b]$ where $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

The proofs of the next two results are almost identical to those for the Henstock integral. We state these theorems for completeness.

THEOREM 6: The function $f : [a, b] \to R$ is $\mathcal{S}$–Henstock integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $|f(\mathcal{P}_1) - f(\mathcal{P}_2)| < \epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are $\mathcal{S}$–subordinate to δ on $[a, b]$.

THEOREM 7: Let $f : [a, b] \to R$.

(a) If f is $\mathcal{S}$–Henstock integrable on $[a,b]$, then f is $\mathcal{S}$–Henstock integrable on every subinterval of $[a,b]$.

(b) If f is $\mathcal{S}$–Henstock integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is $\mathcal{S}$–Henstock integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Suppose that $f : [a,b] \to R$ is $\mathcal{S}$–Henstock integrable on $[a,b]$. By the previous theorem, the function $F(x) = \int_a^x f$ is well-defined on $[a,b]$. We next examine the properties of the function F. The following version of Henstock's Lemma is valid.

LEMMA 8: (Henstock's Lemma) Let $f : [a,b] \to R$ be $\mathcal{S}$–Henstock integrable on $[a,b]$ and let $F(x) = \int_a^x f$. Given $\epsilon > 0$, choose a positive function δ on $[a,b]$ so that $|f(\mathcal{P}) - \int_a^b f| < \epsilon$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a,b]$. If $\mathcal{P}_1 = \{(s_i,[c_i,d_i]) : 1 \le i \le N\}$ is $\mathcal{S}$–subordinate to δ, then

$$|f(\mathcal{P}_1) - F(\mathcal{P}_1)| \le \epsilon \quad \text{and} \quad \sum_{i=1}^{N} |f(s_i)(d_i - c_i) - (F(d_i) - F(c_i))| \le 2\epsilon.$$

DEFINITION 9: Let $F : [a,b] \to R$ and let $x \in [a,b]$. The function F is $\mathcal{S}$–continuous at x if for each $\epsilon > 0$ there exists $\eta > 0$ such that $|F(t) - F(x)| < \epsilon$ whenever $|t - x| < \eta$ and $t \in S_x$. The function F is $\mathcal{S}$–differentiable at x if $\lim\limits_{\substack{t \to x \\ t \in S_x}} \dfrac{F(t) - F(x)}{t - x}$ exists. We will use $F'_{\mathcal{S}}(x)$ to denote the $\mathcal{S}$–derivative of F at x.

Due to the nature of $\mathcal{S}$, we see that F is approximately continuous at x if it is $\mathcal{S}$–continuous at x. Furthermore, if F is approximately continuous on $[a,b]$, then there exists a distribution $\mathcal{S}$ on $[a,b]$ such that F is $\mathcal{S}$–continuous on $[a,b]$. A similar statement holds for approximate derivatives and $\mathcal{S}$–derivatives. It is clear that F is $\mathcal{S}$–continuous at x if it is $\mathcal{S}$–differentiable at x.

THEOREM 10: If $f : [a,b] \to R$ is $\mathcal{S}$–Henstock integrable on $[a,b]$, then the function $F(x) = \int_a^x f$ is $\mathcal{S}$–continuous on $[a,b]$.

PROOF: Let $x \in [a,b]$ and let $\epsilon > 0$. Choose a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - \int_a^b f| < \epsilon/2$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a,b]$ and let $\eta = \min\{\delta(x), \epsilon/2(1 + |f(x)|)\}$. Let $t \in S_x$ with $|t - x| < \eta$. Since the interval with endpoints t, x and tag x is $\mathcal{S}$–subordinate to δ, we can use Henstock's Lemma to obtain

$$|F(t) - F(x)| \le |F(t) - F(x) - f(x)(t - x)| + |f(x)(t - x)| < \epsilon/2 + |f(x)|\eta < \epsilon.$$

Therefore, the function F is $\mathcal{S}$–continuous at x.

THEOREM 11: **If $f : [a,b] \to R$ is $\mathcal{S}$–Henstock integrable on $[a,b]$, then the function $F(x) = \int_a^x f$ is $\mathcal{S}$–differentiable almost everywhere on $[a,b]$ and $F'_{\mathcal{S}} = f$ almost everywhere on $[a,b]$.**

PROOF: Let A^+ be the set of all points x in $[a,b)$ such that either $\displaystyle\lim_{\substack{t \to x^+ \\ t \in S_x}} \frac{F(t) - F(x)}{t - x}$ does not exist or does not equal $f(x)$. For each x in A^+, there exists $\eta_x > 0$ with the following property: for each $h > 0$ there exists a point x_h in $S_x \cap (x, x+h)$ such that

$$|F(x_h) - F(x) - f(x)(x_h - x)| \geq \eta_x(x_h - x).$$

Let $A_n^+ = \{x \in A^+ : \eta_x \geq 1/n\}$. We will show that $\mu^*(A_n^+) = 0$ for each positive integer n.

Fix n and let $\epsilon > 0$. Choose a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - \int_a^b f| < \epsilon/4n$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a,b]$. The collection of intervals $\mathcal{I} = \{[x, x_h] : x \in A_n^+, 0 < h < \delta(x)\}$ forms a Vitali covering of A_n^+. By the Vitali Covering Lemma, there exists a finite collection $\{[c_i, d_i] : 1 \leq i \leq N\}$ of disjoint intervals in $\mathcal{I}$ such that $\mu^*(A_n^+) \leq \sum_i (d_i - c_i) + \epsilon/2$. Note that each $(c_i, [c_i, d_i])$ is $\mathcal{S}$–subordinate to δ and that

$$(d_i - c_i)\eta_{c_i} \leq |F(d_i) - F(c_i) - f(c_i)(d_i - c_i)|.$$

Using Henstock's Lemma, we obtain

$$\begin{aligned}
\sum_{i=1}^{N}(d_i - c_i) &\leq \sum_{i=1}^{N} \frac{1}{\eta_{c_i}} |F(d_i) - F(c_i) - f(c_i)(d_i - c_i)| \\
&\leq n \sum_{i=1}^{N} |f(c_i)(d_i - c_i) - (F(d_i) - F(c_i))| \\
&\leq n\, 2\epsilon/4n \\
&= \epsilon/2.
\end{aligned}$$

It follows that $\mu^*(A_n^+) \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $\mu^*(A_n^+) = 0$.

Since $A^+ = \cup_n A_n^+$, the set A^+ has measure zero. In an analogous manner, we can show that $\mu^*(A^-) = 0$, where A^- is the set of all points x in $(a,b]$ such that either $\displaystyle\lim_{\substack{t \to x^- \\ t \in S_x}} \frac{F(t) - F(x)}{t - x}$ does not exist or does not equal $f(x)$. Since the set of all points x in $[a,b]$ for which either $F'_{\mathcal{S}}(x)$ does not exist or $F'_{\mathcal{S}}(x) \neq f(x)$ is contained in $A^+ \cup A^-$, it follows that $F'_{\mathcal{S}} = f$ almost everywhere on $[a,b]$.

COROLLARY 12: **If $f : [a,b] \to R$ is $\mathcal{S}$–Henstock integrable on $[a,b]$, then f is a measurable function.**

PROOF: Let $F(x) = \int_a^x f$. Since F is approximately continuous, it is measurable. Now f is measurable since it is almost everywhere the approximate derivative of a measurable function. See page 299 of Saks [6].

The next theorem shows that the $\mathcal{S}$–Henstock integral recovers a function from its $\mathcal{S}$–derivative and hence from its approximate derivative.

THEOREM 13: Let $F : [a,b] \to R$ be $\mathcal{S}$–continuous and let $f : [a,b] \to R$. If $F'_{\mathcal{S}} = f$ on $[a,b]$ except for a countable set, then f is $\mathcal{S}$–Henstock integrable on $[a,b]$ and $\int_a^x f = F(x) - F(a)$ for all $x \in [a,b]$.

PROOF: Let $E = \{x \in [a,b] : F'_{\mathcal{S}}(x) \neq f(x)\}$ and write $E = \{x_k\}$. Let $\epsilon > 0$. For each k choose $\delta(x_k) > 0$ so that $|f(x_k)|\delta(x_k) < \epsilon/2^k$ and $|F(I)| < \epsilon/2^k$ for each $I \in \mathcal{I}_{x_k}$ that satisfies $\mu(I) < \delta(x_k)$. For each $x \in [a,b] - E$ choose $\delta(x) > 0$ so that $|f(x)\mu(I) - F(I)| < \epsilon\mu(I)$ whenever $I \in \mathcal{I}_x$ and $\mu(I) < \delta(x)$. This defines a positive function δ on $[a,b]$. Suppose that $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a,b]$. Let $\mathcal{P}_E$ be the subset of $\mathcal{P}$ that has tags in E and let $\mathcal{P}_d = \mathcal{P} - \mathcal{P}_E$. For each k, let $\mathcal{P}_k$ be the subset of $\mathcal{P}_E$ that has x_k as a tag. We then have

$$|f(\mathcal{P}_E)| < \sum_k |f(x_k)|\, 2\delta(x_k) < \sum_k 2\epsilon/2^k < 2\epsilon \quad \text{and} \quad |F(\mathcal{P}_E)| \leq \sum_k |F(\mathcal{P}_k)| < \sum_k \epsilon/2^k < \epsilon,$$

and hence

$$|f(\mathcal{P}) - F(\mathcal{P})| \leq |f(\mathcal{P}_d) - F(\mathcal{P}_d)| + |f(\mathcal{P}_E)| + |F(\mathcal{P}_E)| < \epsilon\,(b-a) + 2\epsilon + \epsilon.$$

Therefore the function f is $\mathcal{S}$–Henstock integrable on $[a,b]$ and $\int_a^b f = F(b) - F(a)$. A similar argument shows that $\int_a^x f = F(x) - F(a)$ for all $x \in [a,b]$.

We next give a descriptive characterization of the $\mathcal{S}$–Henstock integral. As is to be expected, this characterization involves a notion of absolute continuity. The next definition provides the generalization that is needed in our case. See Lee [3] for another absolute continuity condition related to this type of integral.

DEFINITION 14: Let $F : [a,b] \to R$ and let $E \subset [a,b]$. The function F is $AC_{\mathcal{S}}$ on E if for each $\epsilon > 0$ there exist a positive number η and a positive function δ on E such that $|F(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ, all of the tags of $\mathcal{P}$ are in E, and $\mu(\mathcal{P}) < \eta$. The function F is $ACG_{\mathcal{S}}$ on E if E can be written as a countable union of sets on each of which the function F is $AC_{\mathcal{S}}$.

It is easy to verify that an $ACG_{\mathcal{S}}$ function on $[a,b]$ is $\mathcal{S}$–continuous on $[a,b]$. We next prove two simple lemmas, then a descriptive characterization of the $\mathcal{S}$–Henstock integral. See Gordon [1] for a similar characterization of the Henstock integral.

LEMMA 15: Suppose that $F:[a,b]\to R$ is $ACG_{\mathcal{S}}$ on $[a,b]$ and let $E\subset[a,b]$. If $\mu(E)=0$, then for each $\epsilon>0$ there exists a positive function δ on E such that $|F(\mathcal{P})|<\epsilon$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and all of the tags of $\mathcal{P}$ are in E.

PROOF: Let $E=\bigcup_n E_n$ where the E_n's are disjoint and F is $AC_{\mathcal{S}}$ on each E_n. Let $\epsilon>0$. For each n, there exist a positive function δ_n on E_n and a positive number η_n such that $|F(\mathcal{P})|<\epsilon/2^n$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ_n, all of the tags of $\mathcal{P}$ are in E_n, and $\mu(\mathcal{P})<\eta_n$. For each n, choose an open set O_n such that $E_n\subset O_n$ and $\mu(O_n)<\eta_n$. Let $\delta(x)=\min\{\delta_n(x),\rho(x,\mathcal{C}O_n)\}$ for $x\in E_n$. Suppose that $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and that all of the tags of $\mathcal{P}$ are in E. Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tags in E_n. Note that $\mu(\mathcal{P}_n)<\eta_n$ and compute $|F(\mathcal{P})|\le\sum_n|F(\mathcal{P}_n)|<\sum_n\epsilon/2^n<\epsilon$. This completes the proof.

LEMMA 16: Suppose that $f:[a,b]\to R$ and let $E\subset[a,b]$. If $\mu(E)=0$, then for each $\epsilon>0$ there exists a positive function δ on E such that $|f(\mathcal{P})|<\epsilon$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and all of the tags of $\mathcal{P}$ are in E.

PROOF: For each positive integer n, let $E_n=\{x\in E: n-1\le|f(x)|<n\}$ and let $\epsilon>0$. For each n, choose an open set O_n such that $E_n\subset O_n$ and $\mu(O_n)<\epsilon/n2^n$. Let $\delta(x)=\rho(x,\mathcal{C}O_n)$ for $x\in E_n$. Suppose that $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and that all of the tags of $\mathcal{P}$ are in E. Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tags in E_n and compute $|f(\mathcal{P})|\le\sum_n|f(\mathcal{P}_n)|<\sum_n n\mu(O_n)<\sum_n\epsilon/2^n<\epsilon$. This completes the proof.

THEOREM 17: A function $f:[a,b]\to R$ is $\mathcal{S}$–Henstock integrable on $[a,b]$ if and only if there exists an $ACG_{\mathcal{S}}$ function F on $[a,b]$ such that $F'_{\mathcal{S}}=f$ almost everywhere on $[a,b]$.

PROOF: Suppose first that f is $\mathcal{S}$–Henstock integrable on $[a,b]$ and let $F(x)=\int_a^x f$. Then $F'_{\mathcal{S}}=f$ almost everywhere on $[a,b]$ by Theorem 11. For each positive integer n, let $E_n=\{x\in[a,b]: n-1\le|f(x)|<n\}$. Fix n and let $\epsilon>0$. Since f is $\mathcal{S}$–Henstock integrable on $[a,b]$, there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P})-F(\mathcal{P})|<\epsilon$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a,b]$. Let $\eta=\epsilon/n$. Suppose that $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ, all of the tags of $\mathcal{P}$ are in E_n, and $\mu(\mathcal{P})<\eta$. Then using Henstock's Lemma, we obtain

$$|F(\mathcal{P})|\le|F(\mathcal{P})-f(\mathcal{P})|+|f(\mathcal{P})|<\epsilon+n\eta=2\epsilon.$$

Hence the function F is $AC_{\mathcal{S}}$ on E_n and it follows that F is $ACG_{\mathcal{S}}$ on $[a,b]$.

Now suppose that there exists an $ACG_{\mathcal{S}}$ function F on $[a,b]$ such that $F'_{\mathcal{S}}=f$ almost everywhere on $[a,b]$. Let $E=\{x\in[a,b]: F'_{\mathcal{S}}(x)\ne f(x)\}$ and let $\epsilon>0$. For each $x\in[a,b]-E$ choose $\delta(x)>0$ so that $|f(x)\mu(I)-F(I)|<\epsilon\mu(I)$ whenever $I\in\mathcal{I}_x$ and $\mu(I)<\delta(x)$. By the previous two lemmas, we can define $\delta(x)>0$ on E so that $|f(\mathcal{P})|<\epsilon$ and $|F(\mathcal{P})|<\epsilon$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and all of the tags of $\mathcal{P}$ are in E. This defines a positive function δ on $[a,b]$.

Suppose that $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ on $[a,b]$. Let $\mathcal{P}_E$ be the subset of $\mathcal{P}$ that has tags in E and let $\mathcal{P}_d = \mathcal{P} - \mathcal{P}_E$. We then have

$$|f(\mathcal{P}) - F(\mathcal{P})| \leq |f(\mathcal{P}_d) - F(\mathcal{P}_d)| + |f(\mathcal{P}_E)| + |F(\mathcal{P}_E)| < \epsilon(b-a) + \epsilon + \epsilon.$$

Therefore, the function f is $\mathcal{S}$–Henstock integrable on $[a,b]$ and $\int_a^b f = F(b) - F(a)$.

We finish this part of the paper by examining some of the properties of $ACG_{\mathcal{S}}$ functions. We begin by defining $BVG_{\mathcal{S}}$ functions and proving, as should be the case, that $ACG_{\mathcal{S}}$ functions are $BVG_{\mathcal{S}}$.

DEFINITION 18: The function $F : [a,b] \to R$ is $BV_{\mathcal{S}}$ on $E \subset [a,b]$ if there exist a positive function δ on E and a positive number M such that $|F(\mathcal{P})| \leq M$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and all of the tags of $\mathcal{P}$ are in E. The function F is $BVG_{\mathcal{S}}$ on E if E can be written as a countable union of sets on each of which F is $BV_{\mathcal{S}}$.

THEOREM 19: Let $F : [a,b] \to R$ and let $E \subset [a,b]$. If F is $AC_{\mathcal{S}}$ on E, then F is $BV_{\mathcal{S}}$ on E. Consequently, an $ACG_{\mathcal{S}}$ function on $[a,b]$ is $BVG_{\mathcal{S}}$ on $[a,b]$.

PROOF: Since F is $\mathcal{S}$–continuous on E, there exists a positive function δ_1 on E such that $|F(I)| < 1$ whenever $I \in \mathcal{I}_x$ and $\mu(I) < \delta_1(x)$. Since F is $AC_{\mathcal{S}}$ on E, there exist a positive function $\delta < \delta_1$ on E and a positive number η such that $|F(\mathcal{P})| < 1$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ, all of the tags of $\mathcal{P}$ are in E, and $\mu(\mathcal{P}) < \eta$. Let N be the least positive integer such that $(b-a)/N < \eta$ and let

$$K_i = \left[a + (i-1)\frac{b-a}{N}, a + i\,\frac{b-a}{N}\right]$$

for $1 \leq i \leq N$. Suppose that $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and all of the tags of $\mathcal{P}$ are in E. For each i, let $\mathcal{P}_i$ be the subset of $\mathcal{P}$ that has intervals in K_i and let $\mathcal{P}_0 = \mathcal{P} - \bigcup_1^N \mathcal{P}_i$. Note that $\mathcal{P}_0$ contains at most $N-1$ intervals and $|F(I)| < 1$ for each of these intervals. We thus have

$$|F(\mathcal{P})| \leq |F(\mathcal{P}_0)| + \sum_{i=1}^{N} |F(\mathcal{P}_i)| < N - 1 + \sum_{i=1}^{N} 1 = 2N - 1.$$

Hence the function F is $BV_{\mathcal{S}}$ on E.

The best case scenario would be for $ACG_{\mathcal{S}}$ functions to be approximately differentiable almost everywhere. This is indeed the case. We need the following theorem which can be found on page 295 of Saks [6].

THEOREM 20: Let $F : [a,b] \to R$ be measurable and let $E \subset [a,b]$. Then at almost every point of E either F is approximately differentiable or $\overline{F}^{+}_{\text{ap}} = \overline{F}^{-}_{\text{ap}} = +\infty$ and $\underline{F}^{+}_{\text{ap}} = \underline{F}^{-}_{\text{ap}} = -\infty$.

THEOREM 21: Let $F : [a,b] \to R$ be measurable and let $E \subset [a,b]$. If F is BV_S on E, then F is approximately differentiable almost everywhere on E. Consequently, if F is ACG_S on $[a,b]$, then F is approximately differentiable almost everywhere on $[a,b]$.

PROOF: Let $A = \{x \in E : \overline{F}^{+}_{\text{ap}}(x) = +\infty\}$. By the above theorem, it is sufficient to prove that $\mu^*(A) = 0$. Suppose that $\mu^*(A) = \alpha > 0$. There exist a positive function δ on E and a positive number M such that $|F(\mathcal{P})| \le M$ whenever $\mathcal{P}$ is $\mathcal{S}$–subordinate to δ and all of the tags of $\mathcal{P}$ are in E. Choose L such that $L\alpha/2 > M$. For each $x \in A$ and each $h > 0$, there exists $x_h \in S_x \cap (x, x+h)$ such that $\dfrac{F(x_h) - F(x)}{x_h - x} > L$. The collection $\mathcal{I} = \bigcup_{x\in A}\{[x, x_h] : 0 < h < \delta(x)\}$ is a Vitali cover of A. Let $\{[c_i, d_i] : 1 \le i \le N\}$ be a collection of disjoint intervals in $\mathcal{I}$ such that $\sum_i (d_i - c_i) > \alpha/2$. The tagged partition $\mathcal{P} = \{(c_i, [c_i, d_i]) : 1 \le i \le N\}$ is $\mathcal{S}$–subordinate to δ and all of the tags of $\mathcal{P}$ are in E. However,

$$F(\mathcal{P}) = \sum_{i=1}^{N} \big(F(d_i) - F(c_i)\big) > \sum_{i=1}^{N} L(d_i - c_i) > L\alpha/2 > M,$$

a contradiction. It follows that $\mu^*(A) = 0$.

The last property we will prove is also a result to be expected. If two ACG_S functions have the same approximate derivative almost everywhere, they should differ by a constant. This fact follows from the next theorem and the uniqueness of the $\mathcal{S}$–Henstock integral.

THEOREM 22: Let $f : [a,b] \to R$.
(a) If $f = 0$ almost everywhere, then f is $\mathcal{S}$–Henstock integrable on $[a,b]$ and $\int_a^b f = 0$.
(b) If $f \ge 0$ on $[a,b]$ and $\mathcal{S}$–Henstock integrable on $[a,b]$, then $\int_a^b f \ge 0$.
(c) If $f \ge 0$ almost everywhere and $\mathcal{S}$–Henstock integrable on $[a,b]$, then $\int_a^b f \ge 0$.

PROOF: Part (a) follows from the fact that the $\mathcal{S}$–Henstock integral includes the Henstock integral and part (b) is obvious. Suppose that $f \ge 0$ almost everywhere on $[a,b]$ and $\mathcal{S}$–Henstock integrable on $[a,b]$. Let $f_1 = \max\{f, 0\}$ and let $f_2 = f - f_1$. Then $f_1 \ge 0$ on $[a,b]$ and $f_2 = 0$ almost everywhere on $[a,b]$. Since f_2 and $f_1 = f - f_2$ are $\mathcal{S}$–Henstock integrable on $[a,b]$, we find that $\int_a^b f = \int_a^b f_1 + \int_a^b f_2 \ge 0$.

THEOREM 23: Let $F : [a,b] \to R$ be ACG_S on $[a,b]$.
(a) If $F'_S = 0$ almost everywhere on $[a,b]$, then F is constant on $[a,b]$.
(b) If $F'_S \ge 0$ almost everywhere on $[a,b]$, then F is nondecreasing on $[a,b]$.

PROOF: Let f be the function that equals F'_S when it exists and equals 0 otherwise. Then by Theorem 17, the function f is $\mathcal{S}$–Henstock integrable on $[a,b]$ and $F(x) - F(a) = \int_a^x f$. The theorem now follows from the previous theorem.

It should be noted that a rather crucial and obvious question has been left unresolved. Is every Denjoy-Khintchine integrable function $\mathcal{S}$-Henstock integrable? This amounts to asking whether or not an ACG function is $ACG_{\mathcal{S}}$.

We now turn to the dyadic derivative of a function and discuss some of its properties. The ideas for this section were generated from papers by Kahane [2] and Pacquement [4]. As the name indicates, the dyadic derivative is tied to the dyadic rational numbers. For the remainder of this paper D will represent the set of dyadic rational numbers in $[0,1]$. An interval with endpoints in D will be called a dyadic interval. Two numbers a and b are said to be consecutive dyadic rational numbers if there exist integers p and n such that $a = p/2^n$ and $b = (p+1)/2^n$. Let $x \in [0,1] - D$. For each positive integer n, let $x_n = p/2^n$ where p is the unique integer satisfying $p/2^n < x < (p+1)/2^n$. Note that $x_n < x < x_n + 2^{-n}$ for all n.

DEFINITION 24: Let $F : D \to R$, let $x \in [0,1] - D$, and let $r \in D$.
(a) The function F is dyadic continuous at x if $\lim_{n\to\infty}(F(x_n + 2^{-n}) - F(x_n)) = 0$. The function F is dyadic continuous at r if $\lim_{n\to\infty} F(r - 2^{-n}) = F(r)$ and $\lim_{n\to\infty} F(r + 2^{-n}) = F(r)$.
(b) The function F has a dyadic derivative at x if the limit

$$\lim_{n\to\infty} \frac{F(x_n + 2^{-n}) - F(x_n)}{2^{-n}}$$

exists. The function F has a dyadic derivative at r if both the limits

$$\lim_{n\to\infty} \frac{F(r - 2^{-n}) - F(r)}{-2^{-n}} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{F(r + 2^{-n}) - F(r)}{2^{-n}}$$

exist and are equal. We will use $F'_d(x)$ to denote the dyadic derivative of F at x.

Here are several obvious statements. A continuous function is dyadic continuous. If F has a derivative at x, then F has a dyadic derivative at x. If F has a dyadic derivative at x, then F is dyadic continuous at x. Only the values of F on D are important, that is, F is countably determined. We present some simple examples, then prove a result on monotonicity.

EXAMPLE 25: (a) Let $F(x) = \chi_Q(x)$, the characteristic function of the rational numbers. For this function $F'_d = 0$ for all points in $[0,1]$, $F'_{ap} = 0$ for all points in $[0,1] - Q$, and F' does not exist for any point in $[0,1]$.

(b) Let $x \in [0,1] - D$ and let E be the closed set $\{x_n\} \cup \{x\} \cup \{x_n + 2^{-n}\}$. Let $F(x_n) = 1/n = F(x_n + 2^{-n})$ for each n and let $F(x) = 0$. Define F to be linear on the intervals contiguous to E. Then F is continuous at x, $F'_d(x) = 0$, and $F'(x)$ does not exist.

(c) Let $F : [0,1] \to R$ be approximately differentiable on $[0,1]$ and suppose that $F'_{ap}(x) \neq 0$ for all $x \in [0,1]$. Define $G : [0,1] \to R$ by $G(x) = F(x)$ for x in $[0,1] - D$ and $G(x) = 0$ for x in D. Then

G is approximately differentiable on $[0,1]-D$ and dyadic differentiable on $[0,1]$, but $G'_d \neq G'_{ap}$ on $[0,1]-D$.

THEOREM 26: Let $F:[0,1]\to R$ be continuous and suppose that F'_d exists at each point of $[0,1]$. If $F'_d > 0$ on $[0,1]$, then F is increasing on $[0,1]$.

PROOF: Since F is continuous on $[0,1]$, it is sufficient to prove that F is increasing on D. Let $s,t\in D$ with $s<t$ and suppose that $F(t)\le F(s)$. Express s and t with the common denominator 2^N. There exist points $s_N, t_N \in [s,t]\cap D$ with denominator 2^N such that $t_N - s_N = 2^{-N}$ and $F(t_N)\le F(s_N)$. There exist points $s_{N+1}, t_{N+1}\in [s_N,t_N]\cap D$ with denominator 2^{N+1} such that $t_{N+1}-s_{N+1} = 2^{-N-1}$ and $F(t_{N+1})\le F(s_{N+1})$. Continue this process to obtain a nested sequence $\{[s_n,t_n]\}_N^\infty$ of closed intervals such that s_n, t_n have denominator 2^n, $t_n - s_n = 2^{-n}$, and $F(t_n)\le F(s_n)$. Let $\{z\} = \bigcap_N^\infty [s_n,t_n]$ and observe that

$$F'_d(z) = \lim_{n\to\infty}\frac{F(t_n)-F(s_n)}{t_n-s_n}\le 0,$$

a contradiction. We conclude that $F(s)<F(t)$.

COROLLARY 27: Let $F:[0,1]\to R$ be continuous and suppose that F'_d exists on $[0,1]$.
(a) If $F'_d\ge 0$ on $[0,1]$, then F is nondecreasing on $[0,1]$.
(b) If $F'_d = 0$ on $[0,1]$, then F is constant on $[0,1]$.

We now define a Henstock type integral that integrates dyadic derivatives. As in the first part of the paper, the key idea is to limit the collection of potential tagged intervals. Since we want to recover dyadic derivatives, we limit ourselves to tagged intervals with dyadic endpoints. For each $x\in[0,1]-D$, let $\mathcal{I}_x$ be the collection of all intervals in $[0,1]$ of the form $[x_n, x_n+2^{-n}]$. For each $x\in D$, let $\mathcal{I}_x$ be the collection of all intervals in $[0,1]$ of the form $[x-2^{-n},x]$, $[x,x+2^{-n}]$, and $[x-2^{-n},x+2^{-n}]$.

DEFINITION 28: Let δ be a positive function defined on $[0,1]$. A collection $\mathcal{P}$ of tagged intervals is dyadic subordinate to δ if $d-c<\delta(x)$ and $[c,d]\in\mathcal{I}_x$ whenever $(x,[c,d])\in\mathcal{P}$. If in addition $\mathcal{P}$ is a partition of the dyadic interval $[a,b]$, then we will write $\mathcal{P}$ is dyadic subordinate to δ on $[a,b]$.

Once again we must prove that such partitions exist. Of course, the best we can hope for is that intervals with dyadic endpoints have partitions that are dyadic subordinate to δ for any positive function δ. The next two results show that such partitions do indeed exist.

THEOREM 29: If δ is a positive function defined on $[0,1]$, then there exists a tagged partition of $[0,1]$ that is dyadic subordinate to δ.

PROOF: Suppose that there is no tagged partition of $[0,1]$ that is dyadic subordinate to δ. Bisect the interval $[0,1]$. Let I_1 be one of the subintervals for which there is no tagged partition of I_1 that is dyadic subordinate to δ. Now bisect the interval I_1 and let I_2 be one of the subintervals for which there is no tagged partition of I_2 that is dyadic subordinate to δ. Continue this process to obtain a nested sequence $\{I_n\} = \{[a_n, b_n]\}$ of closed intervals such that there is no tagged partition of I_n that is dyadic subordinate to δ. Note that a_n and b_n are consecutive dyadic rational numbers with denominator 2^n. Let $\{z\} = \bigcap_n I_n$. If $z \notin D$, then $a_n = z_n$ and $b_n = z_n + 2^{-n}$ for each n. There exists an integer N such that $[a_N, b_N] \subset (z - \delta(z), z + \delta(z))$. Hence $(z, [a_N, b_N])$ is dyadic subordinate to δ, a contradiction. If $z \in D$, then either $z = a_n$ for all $n \geq K$ or $z = b_n$ for all $n \geq K$. In either case, there exists an integer N such that $(z, [a_N, b_N])$ is dyadic subordinate to δ, a contradiction. We conclude that there is a tagged partition of $[0,1]$ that is dyadic subordinate to δ.

COROLLARY 30: Let $a, b \in D$ with $a < b$. If δ is a positive function defined on $[a,b]$, then there exists a tagged partition of $[a,b]$ that is dyadic subordinate to δ.

PROOF: If a and b are consecutive dyadic rational numbers with denominator 2^n, then there is a tagged partition of $[a,b]$ that is dyadic subordinate to δ as in the proof of Theorem 30. Since every interval with dyadic rational endpoints can be decomposed into a finite number of intervals with consecutive dyadic rational numbers as endpoints, the proof is complete.

DEFINITION 31: Let $f : [0,1] \to R$ and let $[a,b]$ be a dyadic interval in $[0,1]$. The function f is dyadic Henstock integrable on $[a,b]$ if there exists a real number α such that for each $\epsilon > 0$ there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - \alpha| < \epsilon$ whenever $\mathcal{P}$ is dyadic subordinate to δ on $[a,b]$. We will write $\alpha = (d) \int_a^b f$.

It is clear that every Henstock integrable function on the dyadic interval $[a,b]$ is dyadic Henstock integrable on $[a,b]$ and that the integrals are equal. The reader can verify that the dyadic Henstock integral satisfies the elementary properties of an integral such as linearity and integrability on dyadic subintervals. Suppose that f is dyadic Henstock integrable on the dyadic interval $[a,b]$. The function F defined by $F(x) = (d) \int_a^x f$ is assumed to have $[a,b] \cap D$ as its domain. The following version of Henstock's Lemma holds.

LEMMA 32: (Henstock's Lemma) Let $f : [0,1] \to R$ be dyadic Henstock integrable on the dyadic interval $[a,b]$ and let $F(x) = (d) \int_a^x f$. Given $\epsilon > 0$, choose a positive function δ on $[a,b]$ so that $|f(\mathcal{P}) - F(b)| < \epsilon$ whenever $\mathcal{P}$ is dyadic subordinate to δ on $[a,b]$. If $\mathcal{P}_1 = \{(s_i, [c_i, d_i]) : 1 \leq i \leq N\}$ is dyadic subordinate to δ, then

$$|f(\mathcal{P}_1) - F(\mathcal{P}_1)| \leq \epsilon \quad \text{and} \quad \sum_{i=1}^{N} \left| f(s_i)(d_i - c_i) - (F(d_i) - F(c_i)) \right| \leq 2\epsilon.$$

We next consider the properties of the indefinite dyadic Henstock integral. The arguments are quite similar to those for the $\mathcal{S}$–Henstock integral.

THEOREM 33: Let $f : [0,1] \to R$ and let $[a,b]$ be a dyadic interval in $[0,1]$. If f is dyadic Henstock integrable on $[a,b]$ and $F(x) = (d)\int_a^x f$, then F is dyadic continuous on $[a,b]$.

PROOF: Let $x \in [a,b]$ and let $\epsilon > 0$. Choose a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - F(\mathcal{P})| < \epsilon/2$ whenever $\mathcal{P}$ is dyadic subordinate to δ on $[a,b]$, and choose a positive integer N such that $2^{-N} < \delta(x)$ and $|f(x)|/2^N < \epsilon/2$. Suppose first that $x \in [a,b] - D$. The tagged interval $(x, [x_n, x_n + 2^{-n}])$ is dyadic subordinate to δ for all $n \geq N$ and using Henstock's Lemma, we obtain

$$|F(x_n + 2^{-n}) - F(x_n)| \leq |F(x_n + 2^{-n}) - F(x_n) - f(x)2^{-n}| + |f(x)|/2^n < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence the function F is dyadic continuous at x. Now suppose that $x \in [a,b] \cap D$. The tagged interval $(x, [x, x + 2^{-n}])$ is dyadic subordinate to δ for all $n \geq N$ and using Henstock's Lemma, we obtain

$$|F(x + 2^{-n}) - F(x)| \leq |F(x + 2^{-n}) - F(x) - f(x)2^{-n}| + |f(x)|/2^n < \epsilon/2 + \epsilon/2 = \epsilon.$$

It follows that $\lim_{n\to\infty} F(x + 2^{-n}) = F(x)$. Similarly $\lim_{n\to\infty} F(x - 2^{-n}) = F(x)$. Hence the function F is dyadic continuous at x.

THEOREM 34: Let $f : [0,1] \to R$ and let $[a,b]$ be a dyadic interval in $[0,1]$. If f is dyadic Henstock integrable on $[a,b]$ and $F(x) = (d)\int_a^x f$, then $F_d' = f$ almost everywhere on $[a,b]$.

PROOF: Let $A = \{x \in [a,b] - D : F_d'(x) \neq f(x)\}$. For each $x \in A$ there exists a positive number η_x with the following property: for each $\beta > 0$ there is an interval $I \in \mathcal{I}_x$ with $\mu(I) < \beta$ such that $|f(x) - F(I)/\mu(I)| \geq \eta_x$. Let $A_n = \{x \in A : \eta_x \geq 1/n\}$. Since $A = \bigcup_n A_n$, it is sufficient to prove that $\mu^*(A_n) = 0$ for each n.

Fix n and let $\epsilon > 0$. Choose a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - F(\mathcal{P})| < \epsilon/4n$ whenever $\mathcal{P}$ is dyadic subordinate to δ on $[a,b]$. The collection of intervals

$$\mathcal{I} = \bigcup_{x\in A_n} \{I : (x,I) \text{ is dyadic subordinate to } \delta \text{ and } |f(x)\mu(I) - F(I)| \geq \eta_x\mu(I)\}$$

forms a Vitali cover of A_n. By the Vitali Covering Lemma, there exists a finite collection $\{I_i : 1 \leq i \leq N\}$ of disjoint intervals in $\mathcal{I}$ such that $\mu^*(A_n) \leq \sum_i \mu(I_i) + \epsilon/2$. For each i, choose $x_i \in A_n$ such that (x_i, I_i) is dyadic subordinate to δ. Using Henstock's Lemma, we obtain

$$\sum_{i=1}^N \mu(I_i) \leq \sum_{i=1}^N |f(x_i)\mu(I_i) - F(I_i)|/\eta_{x_i} \leq n\sum_{i=1}^N |f(x_i)\mu(I_i) - F(I_i)| \leq n(2\epsilon/4n) = \epsilon/2.$$

Hence $\mu^*(A_n) \leq \epsilon$ and it follows that $\mu^*(A_n) = 0$ since $\epsilon > 0$ was arbitrary.

The dyadic Henstock integral integrates all dyadic derivatives and recovers the primitive. The proof of this result and those that follow are so similar to the corresponding results in the first part of the paper that they are omitted.

THEOREM 35: Let $F : [0,1] \to R$ be dyadic continuous and let $f : [0,1] \to R$. If $F_d' = f$ on $[0,1]$ except for a countable set, then f is dyadic Henstock integrable on $[0,1]$ and $(d)\int_0^x f = F(x) - F(0)$ for all $x \in D$.

DEFINITION 36: Let $F : [0,1] \to R$ and let $E \subset [0,1]$. The function F is AC_d on E if for each $\epsilon > 0$ there exist a positive number η and a positive function δ on E such that $|F(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is dyadic subordinate to δ, all of the tags of $\mathcal{P}$ are in E, and $\mu(\mathcal{P}) < \eta$. The function F is ACG_d on E if E can be written as a countable union of sets on each of which the function F is AC_d.

THEOREM 37: A function $f : [0,1] \to R$ is dyadic Henstock integrable on $[0,1]$ if and only if there exists an ACG_d function F on $[0,1]$ such that $F_d' = f$ almost everywhere on $[0,1]$.

THEOREM 38: Let $F : [a,b] \to R$ be ACG_d on $[a,b]$.
(a) If $F_d' = 0$ almost everywhere on $[a,b]$, then F is constant on $[a,b]$.
(b) If $F_d' \geq 0$ almost everywhere on $[a,b]$, then F is nondecreasing on $[a,b]$.

Here are some unanswered questions for this section. Is there a continuous function that has a dyadic derivative at each point, but is not differentiable on an uncountable set? Does an ACG_d function have a dyadic derivative almost everywhere?

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