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T.K.Dutta and S.N.Mukhopadhyay, Department of Mathematics, The University of Burdwan, Burdwan 713104, West Bengal, INDIA

On Riemann summable trigonometric series

1. Introduction : It is known that if a trigonometric series is summable (R, k), $k = 2,3,4$, then its (R, k)-sum is P^k-integrable and the series is a P^k-Fourier series (For $k = 2$ see [5] and for $k = 3$ or 4 see [3]). In the present paper we have introduced three integrals : the R^k-integrals, $k = 2,3,4$, which are more appropriate for (R,k)-summable trigonometric series than the P^k-integrals. Then we have shown that if a trigonometric series is summable (R,k) its (R,k) sum is R^k-integrable. The advantages of these integrals are that they have the power of the first order integral, as in [4,7], and consequently the Euler-Fourier formulae for the coefficients of the trigonometric series can be written in its usual form. We have given the proof by obtaining first a result on formal multiplication for Riemann summable trigonometric series analogous to that for Cesaro summable trigonometric series considered in [6; 13, p.370] which has some importance in itself.

2. Definitions and notation : Let f be a real valued function defined on the closed interval $[a,b]$. Let $x_o \in (a,b)$ and $f(x_o) = \alpha_o$. If there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ depending on x_o but not on h such that

$$f(x_o + h) = \sum_{r=o}^{m} \frac{h^r}{r!} \alpha_r + o(h^m)$$

as $h \to o$, then α_m is called the Peano derivative of order m of f at x_o and is denoted by $f_{(m)}(x_o)$. Taking one sided limit one gets the definition of Peano derivative at the end points of the interval.

For a given integer $s \geq o$ and a number $h > o$ let $x_o \pm \frac{s}{2} h \in (a,b)$. Then the s th central difference $\Delta_s(f;x_o,h)$ of f corresponding to x_o and h is defined by

$$\Delta_s(f;\, x_o,\, h) = \sum_{j=o}^{s} (-1)^j \binom{s}{j} f\left(x_o + \left(\frac{s}{2} - j\right)h\right).$$

The upper Riemann derivate of f at x_o of order s is defined as

$$\overline{RD}^s f(x_o) = \limsup_{h \to o} \frac{\Delta_s(f;\, x_o\,,\, h)}{h^s} .$$

Replacing 'lim sup' by 'lim inf' one gets the definition of the lower Riemann derivate $\underline{RD}^s f(x_o)$. If $\overline{RD}^s f(x_o) = \underline{RD}^s f(x_o)$, the common value is called the Riemann derivative of f at x_o of order s and is denoted by $RD^s f(x_o)$.

We shall write

$$A_o(x) = \frac{1}{2} a_o$$

$$A_n(x) = a_n \cos nx + b_n \sin nx,\ n \geq 1$$

$$B_n(x) = b_n \cos nx - a_n \sin nx,\ n \geq 1 .$$

The upper and lower (R, k) sums of the series

$$\text{(T)} \qquad \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=o}^{\infty} A_n(x)$$

at x_o are defined to be the upper and lower limits of

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} A_n(x).\left(\frac{\sin nh}{nh} \right)^k$$

as $h \to o$. If they are equal and finite then the series (T) is said to be summable (R, k) at x_o to the common value. Let the series (T) be integrated term - by - term k times and let the integrated series converge everywhere to a continuous function ϕ . Then

$$\frac{\Delta_k(\phi;\, x_o, 2h)}{(2h)^k} = \frac{1}{2} a_o + \sum_{n=1}^{\infty} A_n(x).\left(\frac{\sin nh}{nh} \right)^k .$$

Thus $\overline{RD}^k\phi(x_o)$ and $\underline{RD}^k\phi(x_o)$ are the upper and lower (R, k) sums of the series (T) at x_o .

If f has the Darboux property in an interval I, then we shall write $f \in \mathcal{D}$ in I. If E is Lebesgue measurable then $|E|$ will denote the measure of E .

A function f is said to have the property $\overline{R}$ (respectively $\underline{R}$) in an interval $[a,b]$ if for every perfect set $P \subset [a,b]$ there is a portion of P in which f restricted to P is upper (respectively lower) semi-continuous and we write $f \in \overline{R}$ (respectively $f \in \underline{R}$) in $[a,b]$.

Following Zygmund [e.g. see **13**, I, p.53, p.364, p.366] we shall say that two series $\sum_{n=0}^{\infty} u_n(x)$ and $\sum_{n=0}^{\infty} v_n(x)$ are uniformly equi-summable (R, k) if the difference $\sum_{n=0}^{\infty} [u_n(x) - v_n(x)]$ is uniformly summable (R, k) to zero and the series $\sum_{n=0}^{\infty} u_n(x)$ and $\sum_{n=0}^{\infty} v_n(x)$ are uniformly equi-summable (R, k) in the wider sense if $\sum_{n=0}^{\infty} [u_n(x) - v_n(x)]$ is uniformly summable (R, k) (not necessarily to zero).

3. Convexity theorems :

Lemma 3.1. Suppose that

i) f is upper semi-continuous in $[a,b]$

ii) $\overline{RD}^2 f \geqslant 0$ in (a,b) except on an enumerable set $E \subset (a,b)$

iii) $\limsup_{h \to 0+} \dfrac{\Delta_2(f;x,h)}{h} \geqslant 0$ for $x \in E$.

Then f is convex in $[a,b]$.

This is proved in [13, I, p.328, Lemma 3.20]. (In fact the result there is for continuous f but the same proof will suffice for upper semi-continuous f).

Lemma 3.2. Suppose that

i) $f \in \overline{R} \cap \mathscr{D}$ in $[a,b]$

ii) $\overline{RD}^2 f \geqslant 0$ in (a,b) except on an enumerable set $E \subset (a,b)$

iii) $\limsup_{h \to 0+} \frac{\Delta_2(f;x,h)}{h} \geqslant 0$ for $x \in E$.

Then f is continuous and convex in $[a,b]$.

Proof. Let G be the set of all points x in $[a,b]$ such that there is a neighbourhood of x relative to $[a,b]$ in which f is upper semi-continuous. Clearly G is open in $[a,b]$. Set $P = [a,b] \sim G$. Then P is closed. Clearly P cannot have isolated points. For, let $x_0 \in P$ be an isolated point of P. So, if $x_0 \in (a,b)$, there is $\sigma > 0$ such that $(x_0 - \sigma, x_0) \cup (x_0, x_0+\sigma) \subset G$. Since in these intervals f is upper semi-continuous, by Lemma 3.1, f is convex there. Therefore, $\lim_{x \to x_0-} f(x)$ and $\lim_{x \to x_0+} f(x)$ exist and since $f \in \mathscr{D}$ in $[a,b]$, $\lim_{x \to x_0-} f(x) = f(x_0) = \lim_{x \to x_0+} f(x)$. So, $x_0 \in G$, which is a contradiction. Similarly a and b cannot be isolated points of P. Thus P is perfect.

Now we show that P is void. Suppose the contrary. Since $f \in \overline{R}$ in $[a,b]$, there is a portion of P, say $J \cap P \neq \phi$, in which f restricted to P is upper semi-continuous. Now the set $J \sim P$, the complement of P in J, is open (relative to J) and $J \sim P \neq \phi$. For otherwise $J \subset P$ and by the property $\overline{R}$ there is an open interval $J_1 \subset J$ such that f is upper semi-continuous on J_1 which implies $J_1 \subset G$ and so $J_1 = J_1 \cap P \neq \phi$. This is contradiction. If I is any component interval of $J \sim P$ then f is upper semi-continuous in I. By Lemma 3.1 and Darboux property of f it is convex in the closure $\overline{I}$ of I and hence f is continuous in $\overline{I}$. From this we conclude that f is upper semi-continuous in the whole of J. For, let $\xi \in P \cap J$ and $\{\xi_n\}$ be any sequence converging to ξ. If ξ is an isolated point of P from one side and $\{\xi_n\}$ tends to ξ from that side, then since f is continuous in the closure of each component interval of $J \sim P$, $\lim_{n\to\infty} f(\xi_n) = f(\xi)$. If for each n there is a component interval $(s_n, t_n) \subset G$ such that $s_n < \xi_n < t_n$ and $s_n \to \xi$, $t_n \to \xi$, then since f is convex in $[s_n, t_n]$, $f(\xi_n) \leqslant \max\{f(s_n), f(t_n)\}$ and since f is upper semi-continuous on $J \cap P$ relative to P, $\limsup_{n\to\infty} f(\xi_n) \leqslant f(\xi)$. Thus f is upper semi-continuous in J. So $J \cap P$ is void, which is a contradiction. Thus P is void. Therefore f is upper semi-continuous in [a,b] and by Lemma 3.1 f is convex in [a,b]. By the Darboux property of f, it is also continuous in [a,b].

<u>Theorem 3.1</u>. If $f_{(2)}$ exists in [a,b] and $\underline{RD}^4 f \geqslant 0$ everywhere in (a, b) then $f_{(2)}$ is continuous and convex in [a,b].

Proof. We first suppose that $\underline{RD}^4 f > 0$ in (a,b). Let $[c,d] \subset (a,b)$ and

$$(1) \qquad F_n(x) = \frac{\Delta_2(f;\ x,\ 2^{-n})}{(2^{-n})^2}, \qquad x \in [c,d].$$

For $x \in [c,d]$

$$0 < \underline{RD}^4 f(x) = \liminf_{h \to 0} \frac{\Delta_4(f;\ x,\ 2h)}{16h^4}$$

$$= \liminf_{h \to 0} \frac{1}{h^2} \left[\frac{\Delta_2(f;x,4h)}{(4h)^2} - \frac{\Delta_2(f;x,2h)}{(2h)^2} \right].$$

Putting $h = 2^{-n-2}$ we get

$$0 < \underline{RD}^4 f(x) \leqslant \liminf_{n \to \infty} \frac{1}{2^{-2n-4}} \left[\frac{\Delta_2(f;x,2^{-n})}{(2^{-n})^2} - \frac{\Delta_2(f;x,2^{-(n+1)})}{(2^{-(n+1)})^2} \right]$$

i.e., $\quad 0 < \liminf_{n \to \infty} \frac{1}{2^{-2n-4}} [F_n(x) - F_{n+1}(x)]$.

So there is a positive number $N(x)$ such that

$$(2) \qquad F_n(x) > F_{n+1}(x) \quad \text{for} \quad n \geqslant N(x)$$

and hence the sequence $\{F_n(x)\}$ is quasi-nonincreasing in $[c,d]$ (for definition see [**10**]). Since $f_{(2)}$ exist, f is continuous in $[a,b]$ and hence F_n is continuous in $[c,d]$ for each n. From (1)

$$(3) \qquad \lim_{n \to \infty} F_n(x) = f_{(2)}(x) \quad \text{for all } x \in [c,d].$$

Therefore, by a Lemma of Saks [**10**, §3] $f_{(2)} \in \overline{R}$ in $[c,d]$.

From (2) and (3) we have

(4) $F_n(x) > f_{(2)}(x)$ for $n \geqslant N(x)$.

Now integrating $\Delta_2(f_{(2)};\ x,t)$ twice, the first being in the CP-sense [1] and the second in the D^*-sense, we have

$$\int_0^{2^{-n}} \int_0^{h} \Delta_2(f_{(2)};x,t)dt\, dh = \int_0^{2^{-n}} [f_{(1)}(x+h)-f_{(1)}(x-h)-2hf_{(2)}(x)]dh$$

$$= f(x+2^{-n})+f(x-2^{-n})-2f(x)-2^{2n}f_{(2)}(x).$$

So, by (1) and (4)

$$\int_0^{2^{-n}} \int_0^{h} \Delta_2(f_{(2)};\ x,t)dt\, dh > 0 \quad \text{for} \quad n \geqslant N(x).$$

Hence $\overline{D}^2 f_{(2)}(x) \geqslant 0$ for $x \in [c,d]$. Since $f_{(2)} \in \mathscr{D}$ in $[c,d]$ [9], by Lemma 3.2 $f_{(2)}$ is continuous and convex in $[c,d]$. Since $[c,d] \subset (a,b)$ is arbitrary and $f_{(2)} \in \mathscr{D}$ in $[a,b]$, $f_{(2)}$ is continuous and convex in $[a,b]$. Thus the theorem is proved for the special case when $\underline{RD}^4 f > 0$ in (a,b).

To complete the proof consider

$$g(n,\ x) = f(x) + \frac{1}{n} \cdot \frac{x^4}{4!}$$

where n is a positive integer. Then $g_{(2)}(n,x)$ exists in $[a,b]$ and $\underline{RD}^4 g = \underline{RD}^4 f + \frac{1}{n} > 0$ in (a,b). Hence, by the special case, $g_{(2)}(n,x)$ is continuous and convex in $[a,b]$ and since $f_{(2)}(x)$ is the uniform limit of $g_{(2)}(n,x)$ as $n \to \infty$, $f_{(2)}(x)$

is also continuous and convex in $[a, b]$.

<u>Lemma 3.3</u>. For every set $E \subset (a,b)$ of measure zero and every $\varepsilon > 0$ there is a function J such that J''' exists, is non-decreasing, non-negative and continuous in $[a,b]$ and

$$RD^4 J(x) = \infty \text{ for } x \in E, \quad J'''(a) = 0 \text{ and } J'''(b) < \varepsilon .$$

<u>Proof</u> : Let $\varepsilon > 0$ and E be arbitrarily fixed. For each n let G_n be an open set such that $E \subset G_n$, $G_{n+1} \subset G_n$ and $|G_n| < \frac{\varepsilon}{2^n}$. Also let

$$\Psi_n(x) = \int_a^x \int_a^{\xi} \int_a^{\eta} |G_n \cap [a,t]| \, dt \, d\eta \, d\xi .$$

Set

$$J(x) = \sum_{n=1}^{\infty} \Psi_n(x) .$$

Since the function $|G_n \cap [a,t]|$ is continuous, non-decreasing and non-negative, Ψ_n''' exists and possesses these properties. The condition $|G_n| < \frac{\varepsilon}{2^n}$ implies that $\sum_{n=1}^{\infty} \Psi_n'''$ converges uniformly and hence J''' exists and possesses those properties and $J'''(b) < \varepsilon$. Finally, if $x_o \in E$ and N is any positive integer, then for sufficiently small $h > o$, $[x_o - 2h, x_o + 2h] \subset G_N$. Since $|G_n \cap [a,t]|$ is differentiable on G_n with $\frac{d}{dt} |G_n \cap [a,t]| = 1$, we have $\Psi_n^{IV}(x) = 1$ for $x \in G_n$. Hence, for $1 \leqslant r \leqslant N$ and $o \leqslant \alpha \leqslant 2h$ we have

$$\Psi_r(x_o+\alpha) = \int_a^{x_o+\alpha}\int_a^{\xi}\int_a^{\eta} |G_r \cap [a,t]|\,dt\,d\eta\,d\xi$$

$$= A_o(x_o) + A_1(x_o)\alpha + A_2(x_o)\alpha^2 + A_3(x_o)\alpha^3 + \frac{\alpha^4}{4!}$$

where $A_i(x_o)$, $i = 0,1,2,3$ does not depend on α . This is also true for $-2h \leqslant \alpha \leqslant 0$. Hence

$$\frac{\Delta_4(\Psi_r \,;\, x_o \,,\, h)}{h^4} = 1 \,, \quad r = 1,2,\ldots,N \;.$$

Since Ψ_n''' is non-decreasing, Ψ_n is 4-convex [2] and therefore

$$\frac{\Delta_4(J;x_o,h)}{h^4} = \sum_{n=1}^{\infty} \frac{\Delta_4(\Psi_n;x_o,h)}{h^4} \geqslant \sum_{n=1}^{N} \frac{\Delta_4(\Psi_n;x_o,h)}{h^4} = N \;.$$

Hence, $RD^4 J(x_o) = \infty$.

<u>Theorem 3.2</u>. Suppose that

i) $f_{(2)}$ exists everywhere in $[a,b]$

ii) $\underline{RD}^4 f \geqslant 0$ almost everywhere in (a,b)

iii) $\underline{RD}^4 f > -\infty$ everywhere in (a,b).

Then $f_{(2)}$ is continuous and convex in $[a,b]$.

<u>Proof</u> : Let $E_o = \{x:\ x \in (a,b),\ \underline{RD}^4 f(x) < 0\}$. Then $|E_o| = 0$. Let $\{\epsilon_n\}$ be a positive null sequence and let $J(n,x)$ be the function of Lemma 3.3 corresponding to E_o and ϵ_n . Then the function

$f(x) + J(n,x)$ satisfies the hypotheses of Theorem 3.1. Hence $f_{(2)}(x) + J_{(2)}(n,x)$ is continuous and convex in $[a,b]$. Since $J_{(2)}(n,x) < (b-a)\,.\,\varepsilon_n$, we have

$$f_{(2)}(x) = \lim_{n\to\infty} \left[f_{(2)}(x) + J_{(2)}(n,x)\right] .$$

Therefore $f_{(2)}$ is convex in $[a,b]$. The continuity of $f_{(2)}$ follows from the Darboux property of $f_{(2)}$.

Using Riemann derivate of order 3 we have

<u>Theorem 3.3</u>. Suppose that

i) $f_{(1)}$ exists everywhere in $[a,b]$

ii) $\underline{RD}^3 f \geqslant 0$ almost everywhere in (a,b)

iii) $\underline{RD}^3 f > -\infty$ everywhere in (a,b).

Then $f_{(1)}$ is continuous and convex in $[a,b]$.

This sharpens a result of [10], since $f_{(1)}$ is the ordinary first derivative of f.

4. The R^k-integrals, $k = 2,3,4$.

Let f be defined and finite almost everywhere in $[a,b]$ and let $B \subset [a,b]$ be a measurable set of measure $b-a$ with $a,b \in B$. A continuous function Q is said to be a R^k-major function, $k = 2,3,4$, of f in $[a,b]$ if

i) $Q_{(k-2)}$ exists everywhere in $[a,b]$

ii) $Q_{(k-1)}$ exists on B

iii) $Q_{(k-1)}(a) = 0$

iv) $\underline{RD}^k Q \geq f$ almost everywhere in (a,b)

v) $\underline{RD}^k Q > -\infty$ everywhere in (a,b).

A function q is a minor function of f if $-q$ is a major function of $-f$. If $k = 4$ or 3 then it is clear from Theorems 3.2 or 3.3 that $Q_{(k-2)} - q_{(k-2)}$ is continuous and convex for every pair of major and minor functions Q and q. If $k = 2$ (the condition (i) is redundant here) then clearly $Q-q$ is continuous and convex (cf. [4, Theorem 1.1]). Hence for $k = 2,3,4$, the function $Q-q$ is k-convex and since $Q_{(k-1)} - q_{(k-1)}$ exists on B, $Q_{(k-1)} - q_{(k-1)}$ is increasing on B [2]. So, by (iii) we have $Q_{(k-1)}(b) \geq q_{(k-1)}(b)$. This being true for every pair Q and q

$$\inf_{\{Q\}} Q_{(k-1)}(b) \geq \sup_{\{q\}} q_{(k-1)}(b). \tag{5}$$

If equality holds in (5) with equal value being finite then f is said to be R^k-integrable in $[a,b]$ with basis B and we write

$$F(b) = (R^k, B) \int_a^b f(t)dt .$$

The following is clear:

Theorem 4.1. i) If f is R^k-integrable in $[a,b]$ with basis B and $x \in B$ then f is R^k-integrable in $[a,x]$ with basis $B \cap [a,x]$ and

$$F(x) = (R^k, B \cap [a,x]) \int_a^x f(t)dt .$$

(ii) The class of all R^k-integrable functions in $[a,b]$ with basis B is a linear space containing constant functions.

(iii) If f is R^k-integrable in $[a,b]$ with basis B and f and g are almost everywhere equal then g is also R^k-integrable and

$$(R^k,B)\int_a^b f(t)dt = (R^k,B)\int_a^b g(t)dt .$$

Theorem 4.2. Let G be continuous in $[a,b]$ and let

i) $G_{(k-2)}$ exist everywhere in $[a,b]$

ii) $-\infty < \underline{RD}^k G \leqslant \overline{RD}^k G < \infty$ everywhere in (a,b).

Then $RD^k G$ exists almost everywhere in (a,b) and there is a $B\subset(a,b)$ such that $G_{(k-1)}$ exists on B where $|B| = b-a$ and for $a_1,b_1\in B$ with $a_1 < b_1$, $RD^k G$ is R^k-integrable with basis $B\cap[a_1,b_1]$ and

$$(6)\qquad (R^k,B\cap[a_1,b_1])\int_{a_1}^{b_1} RD^k G(t)dt = G_{(k-1)}(b_1) - G_{(k-1)}(a_1) .$$

Proof : Since

$$-\infty < \underline{RD}^k G \leqslant \overline{RD}^k G < \infty \quad \text{everywhere in } (a,b)$$

by [8, Theorem 1] $RD^k G$ and $G_{(k-1)}$ exist almost everywhere in (a,b). Let $G_{(k-1)}$ exist on $B\subset(a,b)$ where $|B| = b-a$ and let $a_1,b_1\in B$. Then

$$G(x) - G_{(k-1)}(a_1)\cdot\frac{x^{k-1}}{(k-1)!}$$

is both a major and a minor function of $RD^k G$ in $[a_1, b_1]$. Hence $RD^k G$ is R^k-integrable in $[a_1, b_1]$ with basis $B \cap [a_1, b_1]$ and the relation (6) holds.

5. Formal multiplication of Riemann summable trigonometric series and applications

The formal product of the series

$$(7) \qquad \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=o}^{\infty} A_n(x)$$

and $g(x) = \lambda \cos px + \mu \sin px$ where λ and μ are constants and p is a fixed positive integer, is the series obtained by multiplying each term of (7) by $g(x)$, replacing the trigonometric products by sums of sines and cosines and then rearranging the terms in the form

$$(8) \qquad \frac{1}{2} u_o + \sum_{n=1}^{\infty} (u_n \cos nx + v_n \sin nx) = \sum_{n=o}^{\infty} U_n(x)$$

say, where

$$u_o = \lambda a_p + \mu b_p$$

$$(9) \qquad u_n = \frac{1}{2} [\lambda(a_{n-p} + a_{n+p}) - \mu(b_{n-p} - b_{n+p})] , \qquad n \geqslant 1$$

$$v_n = \frac{1}{2} [\lambda(b_{n-p} + b_{n+p}) + \mu(a_{n-p} - a_{n+p})] , \qquad n \geqslant 1$$

with the convention that $a_{-s} = a_s$, $b_{-s} = - b_s$.

Let the series conjugate to the series (8) be denoted by

(10) $$\sum_{n=1}^{\infty}(-v_n \cos nx + u_n \sin nx) = -\sum_{n=1}^{\infty} V_n(x) .$$

The series conjugate to the series (7) is

(11) $$\sum_{n=1}^{\infty}(-b_n \cos nx + a_n \sin nx) = -\sum_{n=1}^{\infty} B_n(x) .$$

The formal product of (11) and $g(x)$ can be obtained as above replacing a_o by 0, a_n by $-b_n$ and b_n by a_n for $n \geqslant 1$ in (7) and noting that b_o is also zero. Let the formal product of (11) and $g(x)$ be

$$\frac{1}{2} c_o + \sum_{n=1}^{\infty}(c_n \cos nx + d_n \sin nx) = \sum_{n=o}^{\infty} C_n(x)$$

where

$$c_o = -\lambda b_p + \mu a_p$$

$$c_n = \frac{1}{2}[-\lambda(b_{n-p} + b_{n+p}) - \mu(a_{n-p} - a_{n+p})], \quad n \geqslant 1$$

$$d_n = \frac{1}{2}[\lambda(a_{n-p} + a_{n+p}) - \mu(b_{n-p} - b_{n+p})], \quad n \geqslant 1 .$$

Here, of course, $b_{-s} = b_s$, $a_o = 0$, $a_{-s} = -a_s$.

Therefore

$$c_n = -v_n \quad \text{for} \quad n > p$$

and

$$d_n = u_n \quad \text{for} \quad n > p .$$

Thus we get

<u>Theorem 5.1</u>. The series conjugate to the formal product of a trigonometric series and $g(x) = \lambda \cos px + \mu \sin px$, p being a fixed positive integer and λ, μ being constants, is the formal product of the series conjugate to the given trigonometric series and $g(x)$ plus a trigonometric polynomial of order at most p.

The following result is analogous to that in [6; 13, I, p. 370].

<u>Theorem 5.2</u>. Suppose $a_n = o(n^{\alpha}) = b_n$, $o < \alpha$. Then for each positive integer $m > \alpha + 1$

$$\sum_{n=o}^{\infty} U_n(x) \quad \text{and} \quad \sum_{n=o}^{\infty} A_n(x).g(x)$$

are uniformly equi-summable (R, m); and

$$\sum_{n=1}^{\infty} V_n(x) \quad \text{and} \quad \sum_{n=1}^{\infty} B_n(x).g(x)$$

are uniformly equi-summable (R,m) in the wider sense.

<u>Proof</u> : We suppose that m is even and $m = 2k$. The proof for odd m is similar. We have

$$(12) \quad (a_n \cos nx + b_n \sin nx)(\lambda \cos px + \mu \sin px)$$

$$= \tfrac{1}{2}[(\lambda a_n - \mu b_n) \cos(n+p)x + (\lambda a_n + \mu b_n)\cos(n-p)x$$

$$+ (\lambda b_n + \mu a_n)\sin(n+p)x + (\lambda b_n - \mu a_n)\sin(n-p)x]$$

and hence from (7) and (12)

$$(13)\quad \sum_{n=o}^{\infty} A_n(x).g(x) = \frac{1}{2} a_o(\lambda \cos px + \mu \sin px)$$

$$+ \frac{1}{2} \sum_{n=1}^{\infty} [(\lambda a_n - \mu b_n)\cos(n+p)x + (\lambda a_n + \mu b_n)\cos(n-p)x$$

$$+ (\lambda b_n + \mu a_n)\sin(n+p)x + (\lambda b_n - \mu a_n)\sin(n-p)x] .$$

Integrating (13) term-by-term 2k times and noting that the series obtained after integration converges absolutely, the right hand side of (13) becomes

$$(14)\quad (-1)^k . \frac{1}{2}[a_o(\lambda \frac{\cos px}{p^{2k}} + \mu \frac{\sin px}{p^{2k}}) + \sum_{n=1}^{\infty} (\lambda a_n - \mu b_n).\frac{\cos(n+p)x}{(n+p)^{2k}}$$

$$+ \sum_{\substack{n=1\\ n\neq p}}^{\infty} (\lambda a_n + \mu b_n).\frac{\cos(n-p)x}{(n-p)^{2k}} + \sum_{n=1}^{\infty} (\lambda b_n + \mu a_n).\frac{\sin(n+p)x}{(n+p)^{2k}}$$

$$+ \sum_{\substack{n=1\\ n\neq p}}^{\infty} (\lambda b_n - \mu a_n).\frac{\sin(n-p)x}{(n-p)^{2k}}] + \frac{1}{2}(\lambda a_p + \mu b_p).\frac{x^{2k}}{(2k)!}$$

$$= \frac{1}{2}(\lambda a_p + \mu b_p).\frac{x^{2k}}{(2k)!}$$

$$+ (-1)^k . \frac{1}{2}[\sum_{n=o}^{\infty} (\lambda a_n - \mu b_n).\frac{\cos(n+p)x}{(n+p)^{2k}} + \sum_{\substack{n=1\\ n\neq p}}^{\infty} (\lambda a_n + \mu b_n).\frac{\cos(n-p)x}{(n-p)^{2k}}$$

$$+ \sum_{n=o}^{\infty} (\lambda b_n + \mu a_n).\frac{\sin(n+p)x}{(n+p)^{2k}} + \sum_{\substack{n=1\\ n\neq p}}^{\infty} (\lambda b_n - \mu a_n).\frac{\sin(n-p)x}{(n-p)^{2k}}] .$$

Now

$$\sum_{n=o}^{\infty} (\lambda a_n - \mu b_n) \cdot \frac{\cos(n+p)x}{(n+p)^{2k}} = \sum_{n=p}^{\infty} (\lambda a_{n-p} - \mu b_{n-p}) \cdot \frac{\cos nx}{n^{2k}} ,$$

$$\sum_{\substack{n=1\\ n\neq p}}^{\infty} (\lambda a_n + \mu b_n) \cdot \frac{\cos(n-p)x}{(n-p)^{2k}}$$

$$= \sum_{n=1}^{p-1} (\lambda a_n + \mu b_n) \cdot \frac{\cos(n-p)x}{(n-p)^{2k}} + \sum_{n=p+1}^{\infty} (\lambda a_n + \mu b_n) \cdot \frac{\cos(n-p)x}{(n-p)^{2k}}$$

$$= \sum_{n=1}^{p-1} (\lambda a_{p-n} + \mu b_{p-n}) \cdot \frac{\cos nx}{n^{2k}} + \sum_{n=1}^{\infty} (\lambda a_{n+p} + \mu b_{n+p}) \cdot \frac{\cos nx}{n^{2k}} ,$$

$$\sum_{n=o}^{\infty} (\lambda b_n + \mu a_n) \cdot \frac{\sin(n+p)x}{(n+p)^{2k}} = \sum_{n=p}^{\infty} (\lambda b_{n-p} + \mu a_{n-p}) \cdot \frac{\sin nx}{n^{2k}}$$

and

$$\sum_{\substack{n=1\\ n\neq p}}^{\infty} (\lambda b_n - \mu a_n) \cdot \frac{\sin(n-p)x}{(n-p)^{2k}}$$

$$= \sum_{n=1}^{p-1} (\lambda b_n - \mu a_n) \cdot \frac{\sin(n-p)x}{(n-p)^{2k}} + \sum_{n=p+1}^{\infty} (\lambda b_n - \mu a_n) \cdot \frac{\sin(n-p)x}{(n-p)^{2k}}$$

$$= \sum_{n=1}^{p-1} (-\lambda b_{p-n} - \mu a_{p-n}) \cdot \frac{\sin nx}{n^{2k}} + \sum_{n=1}^{\infty} (\lambda b_{n+p} - \mu a_{n+p}) \cdot \frac{\sin nx}{n^{2k}} .$$

Since, $a_{-s}=a_s$, $b_{-s}=-b_s$, (14) reduces to

$$\frac{1}{2} (\lambda a_p + \mu b_p) \cdot \frac{x^{2k}}{(2k)}$$

$$+ (-1)^k \sum_{n=1}^{\infty} [\ \frac{1}{2}\left\{\lambda(a_{n-p} + a_{n+p}) - \mu(b_{n-p} - b_{n+p})\right\} \cdot \frac{\cos nx}{n^{2k}}$$

$$+ \frac{1}{2}\left\{\lambda(b_{n-p} + b_{n+p}) + \mu(a_{n-p} - a_{n+p})\right\} \cdot \frac{\sin nx}{n^{2k}}\ ]$$

and this is by (9)

$$= \frac{1}{2} u_o \cdot \frac{x^{2k}}{(2k)!} + (-1)^k \sum_{n=1}^{\infty} (u_n \cos nx + v_n \sin nx)/n^{2k} .$$

So, integrating

$$(15) \qquad \sum_{n=o}^{\infty} [U_n(x) - A_n(x).g(x)]$$

term-by-term 2k-times we get zero for all x. Therefore (15) is uniformly summable (R, 2k) to zero. Hence, the first part of the theorem is proved.

For the second part, proceeding as above with the series $\sum_{n=1}^{\infty} B_n(x)$ and noting Theorem 5.1 we can conclude that integrating

$$\sum_{n=1}^{\infty} [V_n(x) - B_n(x).g(x)]$$

term-by-term 2k-times we get a trigonometric polynomial of order at most p and hence the result follows.

<u>Corollary 5.1</u>. Under the hypotheses of Theorem 5.2 if the upper and lower (R,m) sums of $\sum_{n=o}^{\infty} A_n(x)$ at x_o are $\overline{f}(x_o)$ and $\underline{f}(x_o)$ respectively then the upper and lower (R,m) sums of $\sum_{n=o}^{\infty} U_n(x)$ are $\overline{f}(x_o).g(x_o)$ and $\underline{f}(x_o).g(x_o)$ respectively when $g(x_o) > 0$ and are $\underline{f}(x_o).g(x_o)$ and $\overline{f}(x_o).g(x_o)$ respectively when $g(x_o) < 0$.

Proof : It is sufficient to note that the upper and lower (R,m) sums of $\sum_{n=0}^{\infty} A_n(x).g(x)$ at x_0 are $\overline{f}(x_0).g(x_0)$ and $\underline{f}(x_0).g(x_0)$ respectively when $g(x_0) > 0$ and are $\underline{f}(x_0).g(x_0)$ and $\overline{f}(x_0).g(x_0)$ respectively when $g(x_0) < 0$ and that if the upper and lower (R,m) sums of $\sum_{n=0}^{\infty} E_n(x)$ at x_0 are $\overline{e}(x_0)$ and $\underline{e}(x_0)$ respectively and if $\sum_{n=0}^{\infty} D_n(x)$ is summable (R,m) to 0 at x_0 then the upper and lower (R,m) sums of $\sum_{n=0}^{\infty} [E_n(x) + D_n(x)]$ at x_0 are $\overline{e}(x_0)$ and $\underline{e}(x_0)$ respectively.

Lemma 5.1. If $a_n = o(n) = b_n$ and if the upper and lower (R,3) or (R,4) sums of (7) are finite at x_0 then the series $\sum_{n=1}^{\infty} A_n(x)/n^2$ converges.

This is due to Verblunsky [11 , Lemma 7; also 12].

Theorem 5.3. Let $a_n = o(n) = b_n$ and let the series (7) have finite upper and lower (R,k) sums everywhere (k = 3 or 4). Then (7) is almost everywhere summable (R,k) to a function say $f(x)$ and there is a periodic set C of period 2π and of full measure such that for $\alpha \in C$, the functions $f(x)$, $f(x).\cos nx$ and $f(x).\sin nx$ are R^k-integrable in $[\alpha, \alpha + 2\pi]$ with basis $B = [\alpha, \alpha + 2\pi] \cap C$ and

$$a_n = \frac{1}{\pi}(R^k, B) \int_{\alpha}^{\alpha+2\pi} f(x).\cos nx \, dx \quad , \; n = 0,1,2,\ldots$$

$$b_n = \frac{1}{\pi}(R^k, B) \int_{\alpha}^{\alpha+2\pi} f(x).\sin nx \, dx \quad , \; n = 1,2, \ldots \quad .$$

Moreover, if $a_n = o(n^{\alpha}) = b_n$, $o < \alpha < 1$, then the result is also true for $k = 2$.

Proof : Let $k = 4$. Let

$$\phi(x) = \sum_{n=1}^{\infty} \frac{A_n(x)}{n^4} .$$

Since (7) has finite upper and lower (R, 4) sums, $\overline{RD}^4 \phi$ and $\underline{RD}^4 \phi$ are finite everywhere. So, by [8, Theorem 1] $RD^4 \phi$ and $\phi_{(3)}$ exist almost everywhere. Therefore (7) is summable (R, 4) almost everywhere and $\sum_{n=1}^{\infty} \frac{B_n(x)}{n}$ is summable (R, 3) almost everywhere. So

$$f(x) = \frac{1}{2} a_o + RD^4\phi(x) \text{ , where } RD^4\phi(x) \text{ exists.}$$

Let C_o be the set of points where $\phi_{(3)}$ exists, that is, where $\sum_{n=1}^{\infty} \frac{B_n(x)}{n}$ is summable (R, 3). Then C_o is periodic of period 2π and of full measure.

Now consider the formal product $\sum_{n=o}^{\infty} U_n(x)$ of the series (7) and $g(x) = \lambda \cos px + \mu \sin px$, where p is a fixed positive integer, λ, μ are constants, as defined in (8) and (9). Since $a_n = o(n) = b_n$ and (7) has finite upper and lower (R, 4) sums everywhere, by Corollary 5.1 $\sum_{n=o}^{\infty} U_n(x)$ has finite upper and lower (R, 4) sums everywhere and $u_n = o(n) = v_n$. Since $\sum_{n=o}^{\infty} A_n(x)$ is almost everywhere summable (R, 4) to $f(x)$, by Corollary 5.1 $\sum_{n=o}^{\infty} U_n(x)$ is almost everywhere summable (R, 4) to $f(x).g(x)$. As above there is a periodic set say C_p of period 2π and of full measure where $\sum_{n=1}^{\infty} \frac{V_n(x)}{n}$ is summable (R, 3). Let $C = \bigcap_{p=o}^{\infty} C_p$. Then C is periodic of period 2π and of full measure.

By Lemma 5.1, the series $\sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}$ is convergent everywhere. Let

$$G(x) = - \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2} , \quad H(x) = \sum_{n=1}^{\infty} \frac{B_n(x)}{n^3} .$$

Since $a_n = o(n) = b_n$, by [13, I, p.332, Theorem 2.6] $D^1 H = G$ and by [13, I, p.320, Theorem 2.8] H is smooth. Hence the symmetric derivative $D^1 H$ is the ordinary derivative H' and so $H' = G$ everywhere. By the same argument $\phi' = H$ everywhere. Hence ϕ'' exists everywhere and equals G. Taking any point $\alpha \in C$ and writing $B = C \cap [\alpha, \alpha + 2\pi]$ the function ϕ is such that ϕ'' exists in $[\alpha, \alpha + 2\pi]$, $\phi_{(3)}$ exists on B and $\overline{RD}^4\phi$, $\underline{RD}^4\phi$ are finite in $(\alpha, \alpha + 2\pi)$. Therefore by Theorem 4.2, $RD^4\phi$ is R^4-integrable in $[\alpha, \alpha + 2\pi]$ with basis B and

$$(R^4, B) \int_{\alpha}^{\alpha+2\pi} RD^4\phi(x)dx = \phi_{(3)}(\alpha + 2\pi) - \phi_{(3)}(\alpha) = 0$$

and hence by Theorem 4.1 (ii) - (iii)

$$(R^4, B) \int_{\alpha}^{\alpha+2\pi} [f(x) - \tfrac{1}{2} a_o]dx = 0$$

i.e.

$$a_o = \frac{1}{\pi} (R^4, B) \int_{\alpha}^{\alpha+2\pi} f(x)dx .$$

To determine a_n and b_n, $n \geqslant 1$, we employ formal multiplication and consider the series $\sum_{n=o}^{\infty} U_n(x)$. Proceeding as above with the series $\sum_{n=o}^{\infty} U_n(x)$ we see that $f(x).g(x)$ is R^4-integrable in $[\alpha, \alpha + 2\pi]$ with the same basis B and

$$u_o = \lambda a_p + \mu b_p = \frac{1}{\pi} (R^4, B) \int_{\alpha}^{\alpha+2\pi} f(x).g(x)dx .$$

Putting $p = n$, $\lambda = 1$, $\mu = 0$

$$a_n = \frac{1}{\pi} (R^4, B) \int_{\alpha}^{\alpha+2\pi} f(x).\cos nx \, dx$$

and putting $p = n$, $\lambda = 0$, $\mu = 1$

$$b_n = \frac{1}{\pi} (R^4, B) \int_{\alpha}^{\alpha+2\pi} f(x).\sin nx \, dx .$$

For $k = 3$, the proof is similar.

If $k = 2$, the function

$$\Psi(x) = \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}$$

is continuous everywhere and proceeding as above the proof can be completed

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