Real Analysis Exchange Vol 16 (1990-91)

J. Bobok, KMaDG SvF SVST, Radlinskeho 11, 813 68 Bratislava, Czechoslovakia.

On non-differentiable measure-preserving functions

1. Introduction

Let M be a collection of all continuous probability measures on the interval $[0,1]$ with support equal to the interval $[0,1]$. For $\mu \in M$ let $C(\mu)$ consist of all continuous μ-preserving functions from $[0,1]$ onto $[0,1]$, i.e. $C(\mu) = \{f:[0,1] \to [0,1], f \text{ is continuous}, \forall A \subseteq [0,1]: \mu(A)=\mu(f^{-1}(A))\}$. In what follows, $C(\mu)$ will be endowed by the uniform metric ρ.

The purpose of this note is to prove the existence of non-differentiable functions in the complete metric space $C(\mu)$. Our main result is Theorem 4, which states a general result analogous to one of V. Jarník [1]. A construction of Besicovitch function preserving the Lebesgue measure λ is also presented.

2. Residual sets in $C(\mu)$

In this section we study residual sets in $C(\mu)$. We start with some auxiliary results.

Proposition A. $C(\mu)$, endowed by the uniform metric ρ, is a complete metric space.

Proposition B.

$$f \in C(\mu) \text{ iff } \forall \left(\frac{r}{2^m}, \frac{s}{2^n}\right): \mu\left(\left(\frac{r}{2^m}, \frac{s}{2^n}\right)\right) = \mu\left(f^{-1}\left(\left(\frac{r}{2^m}, \frac{s}{2^n}\right)\right)\right)$$

The following lemmas will be useful when proving the main results.

<u>Lemma 1</u>. The set of piecewise linear functions is dense in $C(\lambda)$.

Proof. Fix $f\in C(\lambda)$, $\varepsilon>0$. We shall construct a piecewise linear function $d^{*}\in C(\lambda)$ with local extremes at rational points only, and such that $\rho(d^{*},f)<\varepsilon$.

First of all, we shall construct a piecewise linear function d (maybe $d\notin C(\lambda)$) with local extremes at rational points only, with properties :

(1) $\rho(f,d)<\frac{\varepsilon}{2}$

(2) for the homeomorphism $h(x)=\lambda(d^{-1}([0,x)))$ defined on $[0,1]$ it holds $\rho(h,id)<\frac{\varepsilon}{2}$.

Let n be a positive integer such that $\frac{1}{2^n}<\frac{\varepsilon}{2}$. For $k\in\{0,1,\dots,2^n-1\}$ it holds $f^{-1}(\frac{k}{2^n},\frac{k+1}{2^n}) = \bigcup_m (a^k_m,b^k_m)$ and $\sum_m (b^k_m-a^k_m)=\frac{1}{2^n}$. There exists a positive integer N_k such that

(3) $$\sum_{m=1}^{N_k}(b^k_m-a^k_m) >\frac{1}{2^n}-\frac{\varepsilon}{2^{n+1}} .$$

Define a function $\hat{d}$ for $x\in\bigcup_{m=1}^{N_k}[a^k_m,b^k_m]$ as follows :

if $f(a^k_m)-f(b^k_m)=0$, then

$$\hat{d}(x)=f(a^k_m)+\frac{f(x^k_m)-f(a^k_m)}{x^k_m-a^k_m}(x-a^k_m) , \quad x\in [a^k_m,x^k_m]$$

$$\hat{d}(x)=f(b^k_m)+\frac{f(x^k_m)-f(b^k_m)}{x^k_m-b^k_m}(x-b^k_m) , \quad x\in [x^k_m,b^k_m]$$

where x^k_m is the point in which f attains its extreme on the interval (a^k_m,b^k_m)(since $f\in C(\lambda)$ and $f(a^k_m)-f(b^k_m)=0$, there has to exist at least one such x^k_m);

if $f(a^k_m)-f(b^k_m)=\frac{1}{2^n}$, then

$$\hat{d}(x)=f(a^k_m)+\frac{f(b^k_m)-f(a^k_m)}{b^k_m-a^k_m}(x-a^k_m) , \quad x\in [a^k_m,b^k_m]$$

For the set $M=\bigcup_{k=0}^{2^n-1}\bigcup_{m=1}^{N_k}[a^k_m,b^k_m]$ it holds $\rho(f,\hat{d})<\frac{\varepsilon}{2}$. There exists a function d defined on $[0,1]$ such that $d_{|M}=\hat{d}$, whereby d is continuous piecewise linear function with properties (1),(2) (to obtain (2) use (3)). We may suppose that d has local extremes at rational points only (use a small perturbation). Obviously, using (2) we have $\rho(h\circ d,f)<\rho(h\circ d,d)+\rho(d,f)<\varepsilon$. The function $h\circ d$ is piecewise linear with local extremes at rational points, and since $\lambda(d^{-1}(h^{-1}(A))=\lambda(h(h^{-1}(A)))=\lambda(A)$ we have $h\circ d\in C(\lambda)$. Thus, we put $d^{*}=h\circ d$ and our construction of the function d^{*} is finished.

[ ]

<u>Lemma 2</u>. The set of piecewise monotone functions is dense in $C(\mu)$.

<u>Proof</u>. Let $\{d_j\}_{j=1}^{\infty}$ be a sequence of piecewise linear functions dense in $C(\lambda)$ (see Lemma 1). It is well-known that $g\in C(\mu)$ iff $g=h^{-1}\circ f\circ h$ for some $f\in C(\lambda)$ and homeomorphism h defined by relation $h(x)=\mu([0,x])$ (g,f are topologically conjugated). Hence the sequence $\{h^{-1}\circ d_j\circ h\}_{j=1}^{\infty}$ is dense in $C(\mu)$.

[ ]

For $\alpha\in(0,1)$, a positive integer n and $\beta\in(0,\frac{\alpha(1-\alpha)}{2^n})$ define the function $k_{n,\alpha,\beta}:[0,1]\to[0,1]$ as follows :

$$k_{n,\alpha,\beta}(x)=\begin{cases} 1 \; ; \; x=1,\ x=\frac{2k+1}{2^n}(1-\alpha),\ k\in\langle 0,1,\dots,2^{n-1}-1\rangle \\ 0 \; ; \; x=\frac{2k}{2^n}(1-\alpha),\ k\in\langle 0,1,\dots,2^{n-1}\rangle \\ 1-\alpha \; ; \; x=1-\alpha+2^n\beta,\ x=\frac{2k+1}{2^n}(1-\alpha)\pm\beta,\ k\in\langle 0,1,\dots,2^{n-1}-1\rangle \\ \text{continuously, linearly; otherwise(with a constant slope on the connected component)} \end{cases}$$

Obviously $k_{n,\alpha,\beta}\in C(\lambda)$. For a positive γ define on the interval $[0,\gamma]$ the function $k_{n,\alpha,\beta,\gamma}$ by $k_{n,\alpha,\beta,\gamma}(x)=\gamma k_{n,\alpha,\beta}(\frac{x}{\gamma})$.

Remark. In the following lemma, the function h_μ is defined on $[a,b]$ by $h_\mu(x)=\mu([a,x])$.

<u>Lemma 3</u>. Let $\mu,\nu\in M$, $[a,b]\times[c,d]\subseteq[0,1]^2$, $f:[a,b]\to[c,d]$ is a strictly monotone continuous function such that $f([a,b])=[c,d]$. For $\gamma=h_\mu(b)-h_\mu(a)$ denote $g_n=g_{n,\alpha,\beta}=f\circ h_\mu^{-1}\circ k_{n,\alpha,\beta,\gamma}\circ h_\mu$, $M_{n,k}=\{x\in[a,b];\ \exists\delta\in(0,\frac{1}{k}):\ \frac{g_n(x+\delta)-g_n(x)}{\delta}>k\}$. Further, suppose

that A and $f^{-1}(A)$ are μ-measurable. Then

(4) for every n,α,β : $\mu(f^{-1}(A))=\mu(g_n^{-1}(A))$

(5) for every positive ε and positive integer k there are n,α,β such that

$$\nu(M_{n,k}) > \nu([a,b])-\varepsilon \quad .$$

Proof. Suppose that f is on [a,b] an increasing function (for a decreasing function the proof is similar). Clearly, the property (4) follows from the fact that the function $k_{n,\alpha,\beta,\gamma}$ preserves the Lebesgue measure on $[0,\gamma]$. We shall prove the property (5).

There exists $\omega \in[c,d]$ such that

(6) $$\nu([f^{-1}(\omega),b])<\varepsilon \quad .$$

Put $\alpha=1-\dfrac{h_\mu(f^{-1}(\omega))}{\gamma}$ and consider $g_{n,\alpha,\beta}$ for suitable n,β. If $K_n=\max\{x\in[a,b);\ g_n(x)=d\}$ then obviously $K_n<b$. It is easy to verify that

(7) $$\lim_{n\to\infty} K_n=f^{-1}(\omega), \qquad (\beta\in(0,\frac{\alpha(1-\alpha)}{2^n}))\quad .$$

For the preimage $g_n^{-1}([c,\omega])$ we have

(8) $$(g_n^{-1}([c,\omega])\cap[a,K_n])\supseteq([a,K_n]\setminus\bigcup_{j=1}^{2^n-1}J_j)$$

where J_j are intervals and $\bigcup_{j=1}^{2^n-1}J_j= g_n^{-1}((\omega,d])\cap[a,f^{-1}(\omega)]$. For a fixed n, the properties of ν imply

(9) $$\lim_{\beta\to 0} \nu(\bigcup_{j=1}^{2^n-1}J_j)=0 \quad .$$

From (6),(7),(9) we conclude that for sufficiently large n and sufficiently small β,

(10) $$\nu([a,K_n]\setminus\bigcup_{j=1}^{2^n-1}J_j) > \nu([a,b])-\varepsilon \quad .$$

Thus, (8) and (10) imply that to prove (5) it is sufficient to show that

$$M_{n,k}\supset(g_n^{-1}([c,\omega])\cap[a,K_n])$$

for any k and suitable $n(k)$. Indeed, for every k_1 there exists $n_1(k_1)$ such that for $x\in(g^{-1}_{n_1,\alpha,\beta}([c,\omega])\cap[a,K_{n_1}])$ we can find $y\in(x,x+\frac{1}{k_1})$ for which $g_{n_1}(y)=d$. Then

$$(11) \qquad \frac{g_{n_1}(y)-g_{n_1}(x)}{y-x} > k_1(d-\omega) \quad .$$

Now, if k_1 satisfies the conditions

$$(12) \qquad k_1>k, \quad k_1(d-\omega)>k \quad ,$$

then from (11) and (12) we obtain

$$M_{n_1,k}\supset(g_{n_1,\alpha,\beta}([c,\omega])\cap[a,K_{n_1}])$$

The proof of (5) is complete.

[]

We recall that by a <u>knot point</u> of function f we mean a point x where $D^+f(x)=D^-f(x)=+\infty$ and $D_+f(x)=D_-f(x)=-\infty$. In addition, $C(\mu)$ with the uniform metric ρ is by Proposition A a complete metric space.

<u>Theorem 4</u>. $C(\mu)$-typical function has a knot point at ν-almost every point.

<u>Proof</u>. Denote $M^+(g)=\{x\in[0,1];D^+g(x)=+\infty\}, G^+=\{g\in C(\mu);\nu(M^+(g))=1\}$ (M_+, G_+, M^-, G^-, M_-, G_- analogously). If we put for positive integers p,q

$$E_{p,q}=\{f\in C(\mu);\ \nu(\{x\in[0,1];\forall\delta\in(0,\frac{1}{p}):\frac{f(x+\delta)-f(x)}{\delta}\le p\})\ge\frac{1}{q}\},$$

then $G^+=\bigcap_p\bigcap_q(C(\mu)\setminus E_{p,q})$. We shall show that G^+ is residual in $C(\mu)$.

I. Denote $M_f=\{x\in[0,1];\forall\delta\in(0,\frac{1}{p}):\frac{f(x+\delta)-f(x)}{\delta}\le p\}$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions from $E_{p,q}$, $f_k\to f$ uniformly. It is easy to verify

$$\bigcap_{l\ge o}\overline{\bigcup_{k\ge l}M_{f_k}}\subseteq M_f \ ,\quad \frac{1}{q}\le\nu(\bigcap_{l\ge o}\overline{\bigcup_{k\ge l}M_{f_k}})\le\nu(M_f)$$

,i.e. $f\in E_{p,q}$ and $E_{p,q}$ is closed.

II. Fix $f\in C(\mu)$ and $\varepsilon>0$. By Lemma 2 there exists in $C(\mu)$ a piecewise monotone function g for which

(13) $\rho(f,g)<\frac{\varepsilon}{2}$.

For a positive integer r, consider the partition $0=x_0<..<x_r=1$ of $[0,1]$ such that for every $j\in(1,2,..,r)$ the following conditions are satisfied :

(14) g is on $I_j=[x_{j-1},x_j]$ monotone

(15) $|g(x_{j-1})-g(x_j)|<\frac{\varepsilon}{2}$.

Since $g_{|I_j}$ satisfies the conditions of Lemma 3 we can replace on I_j the function g by $g_{n(j)}=g_{n(j),\alpha,\beta}$(according to Lemma 3) such that

(16) $\nu(\{x\in I_j;\exists\delta\in(0,\frac{1}{p}):\frac{g_{n(j)}(x+\delta)-g_{n(j)}(x)}{\delta}>p\})>\nu(I_j)-\frac{1}{rq}$

Define on [0,1] a function $\hat{g}$ by $\hat{g}_{|I_j}=g_{n(j),\alpha(j),\beta(j)}$. Then from the properties (13),(14) and (15) we conclude $\rho(f,\hat{g})<\varepsilon$, and by (4) we obtain $\hat{g}\in C(\mu)$. Finally, the property (16) implies

$$\nu(\{x\in[0,1];\exists\delta\in(0,\frac{1}{p}):\ \frac{g(x+\delta)-g(x)}{\delta}>p\})>1-\frac{1}{q}\ ,$$

i.e. $\hat{g}\in(C(\mu)\setminus E_{p,q})$ and $E_{p,q}$ is nowhere dense.

Similarly we can show that the sets G_+,G^-,G_- are residual and therefore the set

$$G=G^+\cap G_+\cap G^-\cap G_-=\{g\in C(\mu);\nu(M^+\cap M_+\cap M^-\cap M_-)=1\}$$

is residual as well.

[]

Corollary. $C(\mu)$-typical function maps at least one ν-null set onto [0,1].

Proof. It is easy to see that any level set of any continuous function contains a point which is not a knot point of the function f.

[]

Remark. By means of Theorem 4 it is possible to show that there exists an absolutely continuous measure μ such that

$C(\mu)$ contains a function with a level set of positive Lebesgue measure.

<u>Theorem 5</u>. All level sets of $C(\mu)$-typical function, are ν-null sets.

<u>Proof</u>. Denote $F_n = \{f \in C(\mu); \exists y: \nu(f^{-1}(y)) \geq \frac{1}{n}\}$. It is known that F_n is a closed set (see [2], p.325, 1.1.2.). From Lemma 2 it follows that F_n is nowhere dense for every positive integer n hence the set $F = \bigcup_{n=1}^{\infty} F_n = \{f \in C(\mu); \exists y: \nu(f^{-1}(y)) > 0\}$ is a set of the first category in $C(\mu)$.

[ ]

3. Besicovitch functions in $C(\lambda)$

We recall that by a <u>Besicovitch function</u> we mean a function which has nowhere unilateral derivative (finite or infinite). In 1932 S. Saks[3] proved that the collection of all Besicovitch functions is of the first category in the space $\mathcal{C}$ of all continuous functions on [0,1]. A similar result holds in $C(\mu)$ (using Theorem 4). We shall show that there exist Besicovitch functions in $C(\lambda)$. The following construction of Besicovitch function from $C(\lambda)$ is a slight modification of that in [4].

<u>Construction</u>

Let $k \geq 4$. Let us construct in $[0,\frac{1}{2}]$ a discontinuum

$$D = [0,\tfrac{1}{2}] \setminus L , \quad \text{where } L = \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{2^m-1} r_{m,p} ,$$

and the open intervals $r_{m,p} = (a_{m,p}, b_{m,p})$ are constructed as follows :

(α) $d_{1,1} = [0,\frac{1}{2}]$, $r_{1,1} \subseteq d_{1,1}$, $\lambda(r_{1,1}) = \frac{1}{2k}$,
$b_{1,1}$ is the center of $d_{1,1}$

(β) if $d_{n,1},..,d_{n,2^{n-1}}$ are (from left to right) the intervals of the set $[0,\frac{1}{2}] \setminus \bigcup_{q=1}^{n} \bigcup_{p=1}^{2^{q-1}} r_{q,p}$, then $r_{n,p} \subseteq d_{n,p}$, $b_{n,p}$ is the center of $d_{n,p}$ and $\lambda(r_{n,p}) = \frac{1}{2k^n}$

It is easy to verify that

$$\lambda(L) = \frac{1}{2(k-2)} \quad , \quad \lambda(D) = \frac{k-3}{2(k-2)} \quad .$$

Remark. For k<4 it is impossible to use this method for a construction of discontinuum D.

Define a function $\phi:[0,\frac{1}{2}] \to [0,1]$ by

$$\phi(x) = 2(\frac{k-2}{k-3})\lambda(D \cap (0,x)) \quad .$$

Obviously $\phi(0)=0$, $\phi(\frac{1}{2})=1$, ϕ is continuous, nondecreasing function, constant on every interval $r_{m,p}$, $\phi(r_{m,p}) = \{\frac{2p-1}{2^m}\}$.
Define a function $p:[0,1] \to [0,1]$ by

$$p(x) = \phi(x) \quad , \quad x \in [0,\frac{1}{2}] \quad ,$$
$$p(x) = \phi(1-x) \quad , \quad x \in [\frac{1}{2},1] \quad .$$

The function p and the interval [0,1] form the well-known step triangle [4].
The above described procedure will be called a construction of a step triangle with base [0,1], height 1 and parameter k.

The set $\{(x,p(x)); x \in [0,\frac{1}{2}]\}$ is the left side of triangle, analogously the set $\{(x,p(x)); x \in [\frac{1}{2},1]\}$ is the right side of triangle. Further, put $u_y = \{(x,y); x \in [0,1]\}$ and let $g(f)$ be a graph of a function f. Now, we can construct a function f as follows :

(c_o) construct a step triangle with the base [0,1], height 1 and parameter k ; the sides of the step triangle define a function f_o

(c_n) construct step triangles whose bases are intervals of the set $\bigcup_{p=1}^{2^{n-1}} u_{\frac{2p-1}{2^n}} \cap g(f_{n-1})$, height $\frac{1}{2^n}$ and parameter k (constructed triangles are placed inwards the bigger triangle on whose side have their bases); the union of sides of all so far constructed triangles define a function f_n .

Finally, put $f=\lim_{n\to\infty} f_n$ (obviously $\rho(f_n, f_{n-1})=\frac{1}{2^n}$) .

The function f is Besicovitch function. We shall not prove this fact; it is possible to use a modification of proof presented in [4]. We shall prove

<u>Theorem 6</u>. $f\in C(\lambda)$.

<u>Proof</u>. By Proposition B it is sufficient to show that for any positive integer n and $k\in\{0,1,..2^n-1\}$ we have

(17) $$\lambda(f^{-1}((\frac{k}{2^n},\frac{k+1}{2^n})))=\frac{1}{2^n}$$

From the construction of the function f it is clear that for n,k mentioned above and positive integer s we have

$$f^{-1}_{n+s}((\frac{k}{2^n},\frac{k+1}{2^n}))=f^{-1}_n((\frac{k}{2^n},\frac{k+1}{2^n})) \quad ,$$

i.e. it suffices to verify (17) only for a function f_n . We shall prove it by induction. It is easy to see that $\lambda(f_0^{-1}((0,1)))=1$. Suppose that for every $k\in\{0,1,...,2^n-1\}$ the equality $\lambda(f^{-1}_{n-1}((\frac{k}{2^{n-1}},\frac{k+1}{2^{n-1}}))=\frac{1}{2^{n-1}}$ holds and fix $k_0\in\{0,..,2^{n-1}\}$

Observe that $\frac{u_{2k_0+1}}{2^n}\cap g(f_{n-1})=\bigcup_{k=1}^{m}[u_k,v_k]\times\{\frac{2k_0+1}{2^n}\}$, where m is a suitable positive integer. Let $k\in\{1,...,m\}$ be fixed. There exist $x\in f^{-1}_{n-1}(\frac{2k_0}{2^n})$ and $y\in f^{-1}_{n-1}(\frac{2k_0+2}{2^n})$ such that

$$x_k = \min(x,y) < u_k < v_k < \max(x,y) = y_k$$

$$\emptyset=(x_k,u_k)\cap(f^{-1}_{n-1}(\frac{2k_0}{2^n})\cup f^{-1}_{n-1}(\frac{2k_0+2}{2^n}))$$

$$\emptyset=(v_k,y_k)\cap(f^{-1}_{n-1}(\frac{2k_0}{2^n})\cup f^{-1}_{n-1}(\frac{2k_0+2}{2^n}))$$

For x_k,u_k,v_k,y_k we shall distinguish four cases :

(i) $(x_k+y_k)\frac{1}{2} = u_k \ , \quad f_{n-1}(x_k) < f_{n-1}(y_k)$

(ii) $\quad (x_k+y_k)\frac{1}{2} = u_k \;, \quad f_{n-1}(x_k) > f_{n-1}(y_k)$

(iii) $\quad (x_k+y_k)\frac{1}{2} = v_k \;, \quad f_{n-1}(x_k) < f_{n-1}(y_k)$

(iv) $\quad (x_k+y_k)\frac{1}{2} = v_k \;, \quad f_{n-1}(x_k) > f_{n-1}(y_k) \quad .$

Consider for example (iii)(the others are similar). Obviously a triangle constructed on the base $\{(w,f_{n-1}(w)), w\in[u_k,v_k]\}$ with height $\frac{1}{2^n}$ tends down. Then

$$f_n^{-1}((\frac{2k_o}{2^n},\frac{2k_o+1}{2^n})) \cap [x_k,y_k] = [x_k,v_k] \quad , \text{ i.e.},$$

$$\lambda(f_n^{-1}((\frac{2k_o}{2^n},\frac{2k_o+1}{2^n})) \cap [x_k,y_k]) = v_k-x_k= \frac{1}{2}(y_k-x_k)$$

and since $((x_{k_1},y_{k_1})\cap(x_{k_2},y_{k_2})=\emptyset)$

$$f_n^{-1}((\frac{2k_o}{2^n},\frac{2k_o+1}{2^n}))= \bigcup_{k=1}^{m}(f_n^{-1}((\frac{2k_o}{2^n},\frac{2k_o+1}{2^n})) \cap [x_k,y_k]) \quad ,$$

we have

$$\lambda(f_n^{-1}((\frac{2k_o}{2^n},\frac{2k_o+1}{2^n}))) = \sum_{k=1}^{m}\frac{1}{2}(y_k-x_k) \quad .$$

We can easily verify that

$$f_{n-1}^{-1}((\frac{2k_o}{2^n},\frac{2k_o+2}{2^n})) = \bigcup_{k=1}^{m}(x_k,y_k) \quad ,$$

and, by induction hypothesis,

$$\sum_{k=1}^{m}\frac{1}{2}(y_k-x_k) = \frac{1}{2^n} \quad .$$

The proof of Theorem 6 is complete.

[ ]

References

1. V. Jarník: Über die Differenzierbarkeit stetiger Funktionen, Fund. Math. 21(1933), 48-58.
2. A. M. Bruckner and G. Petruska: Some typical results on bounded Baire 1 function, Acta Math. Hung. 43(3-4)(1984), 325-333.

3. S. Saks: On the functions of Besicovitch in the space of continuous functions, Fund. Math. 19(1932), 211-219.

4. E. D. Pepper: On continuous functions without a derivative, Fund. Math. 12(1928), 244-253.

Received September 26, 1989