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Martin's Axiom implies a stronger version of Blumberg's Theorem

Let **R** be the real line. In 1922, H. Blumberg proved the following theorem: **Blumberg's Theorem** [Bl]: If f:**R**→**R**, then there is a dense subset D of **R** such that f|D is continuous. Here, f|D is the real valued function on D with the subspace topology.

In any such theorem, it is of interest to ask how much the hypothesis can be weakened or the conclusion strengthened. The obvious way to weaken the hypothesis is to allow the domain of f to be some subset of **R** instead of **R**. A set X⊆Y is **categorically dense** in Y if X∩U is of second category in Y for every nonempty open subset U of Y. Trivial modifications in the proof of Blumberg's Theorem then give the following strengthening:

Proposition: If X is a categorically dense subset of **R**, and f:X→**R**, then there is a dense D⊆X such that f|d is continuous.

If every point of X⊆**R** is isolated, then the same result holds trivially. For similar reasons, this is also true if X is scattered (just let D be the set of isolated points of X). However, if X is dense, it is easy to see that the hypothesis cannot be weakened any further, for if X⊆**R** is dense and of first category, partition X into countably many sets X_n, $n<\omega$, each nowhere dense. Let f(x)=n iff $x\in X_n$, and f obviously cannot be continuous on any dense subset. If X is dense and of second category, but not categorically dense, the same trick can be used on X∩I for some interval I, letting f be constant outside I.

Thus, for dense X, X being categorically dense is both necessary and sufficient (at least for subsets of **R**). It is perhaps somewhat surprising that the other main notion of "largeness", namely measure, does not play a role here (Consider a first category set whose complement has Lebesgue measure zero).

Trying to strengthen the conclusion is more complicated, as it turns out to be independent of the usual axioms of Set Theory. In particular, if the Continuum Hypothesis holds, then D cannot even be uncountable because of the following classical theorem of Sierpinski and Zygmund:

Theorem (Sierpinski-Zygmund [SZ]): There is a function $f:\mathbf{R}\to\mathbf{R}$ such that if $X\subseteq\mathbf{R}$ and $|X|=c$, then $f|X$ is not continuous. Here, c is the cardinality of the continuum.

If the Continuum Hypothesis fails, then the Sierpinski-Zygmund function does not rule out the possibility of an uncountable set D. In 1973, J. Shinoda proved that "$|X|=c$" could not be replaced in the Sierpinski-Zygmund Theorem by "X is uncountable":

Theorem (Shinoda [Sh]): Assume MA+¬CH. If $f:X\to\mathbf{R}$, where $X\subseteq\mathbf{R}$ is uncountable, then there is an uncountable $Y\subseteq X$ such that $f|Y$ is continuous.

The set in Shinoda's Theorem was not dense in general, so a natural question is: "Can the set D in the conclusion of Blumberg's Theorem be made uncountably dense?" The main result of this paper is to show that it is consistent that the answer is yes.

1. Definition: Let $X\subseteq\mathbf{R}$, $f:X\to\mathbf{R}$. If $x\in X$, we will say that x is **f-pleasant** (or just **pleasant** if f is obvious from context) if for every $\varepsilon>0$ there is a $\delta>0$ such that $\{y: |f(y)-f(x)|<\varepsilon\}$ is categorically dense in $(x-\delta,x+\delta)$

2. Lemma: Let $X\subseteq\mathbf{R}$, $f:X\to\mathbf{R}$. Then $A=\{x\in X: x \text{ is not pleasant}\}$ is of first category in $\mathbf{R}$.

Proof: Suppose not. For each $x \in A$ pick $\varepsilon(x)>0$ witnessing that x is not pleasant, and pick rational numbers r(x) and s(x) such that $f(x)-\varepsilon(x)<r(x)<f(x)<s(x)<f(x)+\varepsilon(x)$. For each pair (r,s) of rational numbers, let $A(r,s)=\{x\in A: r(x)=r, s(x)=s\}$. Since A is supposedly of second category and there are only countably many pairs (r,s), some A(r,s) must be of second category, and therefore categorically dense in some interval I. Fix such an r and s. Let $x\in I\cap A(r,s)$ and let $\delta>0$ so that $(x-\delta,x+\delta)\subseteq I$. Then $A(r,s)\subseteq\{y: |f(x)-f(y)|<\varepsilon(x)\}$, so $\{y: |f(x)-f(y)<\varepsilon(x)\}$ is categorically dense in $(x-\delta,x+\delta)$, contradicting the definition of $\varepsilon(x)$.

Our main result will be a consequence of MA+¬CH, but we can get by with less without any additional trouble. It is a well known fact that arguments involving forcing and Martin's Axiom do not generally require the antisymmetry property of the relevant partial orderings (see the treatment in [K], for example). Thus, we will define a partially ordered set ("poset") to be a set with a relation which is reflexive and transitive, but not necessarily antisymmetric. A subset S of a poset is called **centered** if every pair of elements of S is compatible. A partially ordered set **P** is called **σ-centered** if it is the union of countably many centered subsets. Since a σ-centered poset obviously satifies the countable chain condition ("ccc"), "MA(κ) for σ-centered posets" is a weakening of MA(κ). We will need the following well known fact:

3. Proposition: Assume MA(κ) for σ-centered posets. Then every subset of **R** of cardinality κ is of first category.

4. Theorem: Assume MA(κ) for σ-centered posets. Let $X\subseteq\mathbf{R}$ be categorically dense in **R**, and let $f:X\to\mathbf{R}$. Then there is a κ-dense set $D\subseteq X$ such that $f|D$ is continuous. Here, if κ is a cardinal, a set is **κ-dense** if it intersects every nonempty open set U in exactly κ elements.

Proof: Without loss of generality, we may assume that every element of X is f-pleasant and that $X\cap\mathbf{Q}=\varnothing$, where **Q** is the set of rational numbers, for if this is not the case, we could make both statements true by throwing away a first category set from X. Thus, we can note that f, considered as a subset of the plane, has the property that if U is any open subset of the plane intersecting f, then $|U\cap f|>\kappa$. To see this, let $(x,y)\in U\cap f$, where U is open in the plane. By pleasantness of x, there are positive δ, ε such that $(x-\delta,x+\delta)\times(y-\varepsilon,y+\varepsilon)\subseteq U$ and the projection of $f\cap((x-\delta,x+\delta)\times(y-\varepsilon,y+\varepsilon))$ onto the first coordinate is categorically dense in $(x-\delta,x+\delta)$, and therefore (by Proposition 3) of cardinality greater than κ. Thus, by induction on $\alpha<\kappa$ we can define a collection $\{X_\alpha:\alpha<\kappa\}$ of pairwise disjoint subsets of X such that for each $\alpha<\kappa$, $\{(x,f(x)): x\in X_\alpha\}$ is a countable dense subset of f. For each $\alpha<\kappa$, let $f_\alpha=f|X_\alpha$. Let $X'=\bigcup\{X_\alpha: \alpha<\kappa\}$, and let $f'=f|X'$. Since each point $x\in X$ is pleasant, we have the following property:

(*) For every $x\in X$ and $\varepsilon>0$ there is a $\delta>0$ such that for every $\alpha<\kappa$, $\{y\in X_\alpha: |f(x)-f(y)|<\varepsilon\}$ is dense in $(x-\delta,x+\delta)$.

To see that this is true, suppose $x\in X$ and $\varepsilon>0$, and let $\delta>0$ be such that the set $\{y\in X: |f(y)-f(x)|<\varepsilon\}$ is categorically dense in $(x-\delta,x+\delta)$, and therefore dense in $(x-\delta,x+\delta)$. Then, since $\{(x,f(x)): x\in X_\alpha\}$ is dense in f (as a subset of the plane), we have that $\{y\in X_\alpha: |f(x)-f(y)|<\varepsilon\}$ is dense in $(x-\delta,x+\delta)$.

Let $\mathbf{Q}^*=\mathbf{Q}\cup\{-\infty,\infty\}$. By a **rational rectangle** we mean any subset of the plane of the form $(a,b)\times(c,d)$ where a, b, c, $d\in\mathbf{Q}^*$, $a<b$, $c<d$. Define the partial ordering $(\mathbf{P},\leq)$ as follows: The set **P** consists of all ordered pairs (A,S) satisfying the following properties:

(1) A is a finite subset of X′.

(2) S is a finite set of rational rectangles.

(3) If $T_1, T_2\in S$, $T_1\neq T_2$, then $\pi(T_1)\cap\pi(T_2)=\emptyset$, where π is projection onto the first coordinate.

(4) For each $\alpha<\kappa$, $\pi(f_\alpha \cap \bigcup S)$ is dense in $\mathbf{R}$.

(5) $\{(x,f(x)): x\in A\}\subseteq\bigcup S$.

We partially order $\mathbf{P}$ by $(A,S)\leq(A',S')$ iff $A'\subseteq A$ and $\bigcup S\subseteq\bigcup S'$. Note that for every finite set S of rational rectangles, the set $\{(A,S): (A,S)\in\mathbf{P}\}$ is centered, so $\mathbf{P}$ is σ-centered, since there are only countably many such S. Thus, MA for σ-centered posets is relevant for $\mathbf{P}$. We now need to define our dense sets.

Let $\{U_n: n\in\omega\}$ be a countable basis of $\mathbf{R}$, and for each $n\in\omega$ and $\alpha<\kappa$ define $D(\alpha,n)=\{(A,S)\in\mathbf{P}: A\cap U_n\cap X_\alpha\neq\varnothing\}$. To see that each $D(\alpha,n)$ is dense, let $(A,S)\in\mathbf{P}$ and note that by (4), $\pi(f_\alpha\cap\bigcup S)\cap U_n\neq\varnothing$. Let $x\in\pi(f_\alpha\cap\bigcup S)\cap U_n$. Then $(A\cup\{x\},S)\in D(\alpha,n)$ and $(A\cup\{x\},S)\leq(A,S)$.

For each $x\in X'$ and each $n\in\omega-\{0\}$, let $D(x,n)=\{(A,S)\in\mathbf{P}$: If $(x,f(x))\in T$ and $T\in S$, then $\pi'(T)$ has length less than $1/n\}$, where π' is projection onto the second coordinate. Then we claim that each $D(x,n)$ is dense, for let $(A,S)\in\mathbf{P}$. Since x is not rational, by (4) we have that $x\in\pi(\bigcup S)$, so let T be such that $x\in\pi(T)$ and $T\in S$. If $(x,f(x))\notin T$, then $(A,S)\in D(x,n)$, so assume $(x,f(x))\in T$. Let $m>2n$ be such that $c<f(x)-1/m<f(x)+1/m<d$, where $T=(a,b)\times(c,d)$. By (*), there is a $\delta>0$ such that for each $\alpha<\kappa$, $\{y\in X_\alpha: |f(x)-f(y)|<1/m\}$ is dense in $(x-\delta,x+\delta)$. Let $u, v, \in\mathbf{Q}$ be such that $x-\delta<u<x<v<x+\delta$, $a<u<v<b$, and $(u,v)\cap A$ is either $\{x\}$ or $\varnothing$ (depending on whether or not x is in A). Let $r, s\in\mathbf{Q}$ be such that $f(x)-1/2n<r<f(x)-1/m$, $f(x)+1/m<s<f(x)+1/2n$, and $c<r<s<d$. Let $S'=(S-\{T\})\cup\{(a,u)\times(c,d), (u,v)\times(r,s), (v,b)\times(c,d)\}$. Then $(A,S')\leq(A,S)$ and $(A,S')\in D(x,n)$.

Thus, we have shown that the $D(\alpha,n)$'s and the $D(x,n)$'s are all dense in $\mathbf{P}$, so since $|(\kappa\times\omega)\cup(X'\times\omega)|=\kappa$, there are only κ-many such $D(\alpha,n)$'s and $D(x,n)$'s, so there is a filter $G\subseteq\mathbf{P}$ meeting all of these dense sets. Let $D=\bigcup\{A$: For some S, $(A,S)\in G\}$. Note that for every $(A,S)\in G$, $f|D\subseteq\bigcup S$. We will see that D satisfies the conclusion of the Theorem.

First, let $n \in \omega$, $\alpha < \kappa$. Then there is an $(A,S) \in G \cap D(\alpha,n)$, so $A \cap U_n \cap X_\alpha \neq \emptyset$ and thus $D \cap U_n \cap X_\alpha \neq \emptyset$ for each $\alpha < \kappa$. Thus D is κ-dense in **R**.

We now need to show that $f|D$ is continuous, so let $x \in D$ and $\varepsilon > 0$. Let $n \in \omega$ be such that $1/n < \varepsilon$ and let $(A,S) \in G \cap D(x,n)$. Since $x \in D$, there is an $(A',S') \in G$ such that $x \in A'$. Now, (A,S) and (A′,S′) must be compatible, so there must be a $T \in S$ such that $(x,f(x)) \in T$, and since $(A,S) \in D(x,n)$, $\pi'(T)$ has length $<1/n<\varepsilon$. Pick δ such that $(x-\delta,x+\delta) \subseteq \pi(T)$. Thus if $y \in D$ and $|x-y|<\delta$, then $(y,f(y)) \in T$, so $|f(x)-f(y)|<\varepsilon$. Thus $f|D$ is continuous at x, and we are done.

Note that since MA(ω) is true in ZFC, the original Blumberg Theorem is a corollary of Theorem 4.

Theorem 4 is also true if **R** is replaced by $\mathbf{R}^n$. In fact, as was pointed out by one of the referees, the theorem is true for any locally compact, separable metric space, since each such space contains a copy of the irrationals as a dense G_δ.

One possible further generalization, suggested to the author by Jack Brown, is the following: Suppose $f:\mathbf{R} \to \mathbf{R}$. Then is it possible to get a categorically dense D so that $f|D$ is continuous? If MA holds, then the answer is clearly no, since in that case a categorically dense D would have to have cardinality c, and the Sierpinski-Zygmund function would be a counterexample. This suggests the following question:

Question: Is it consistent with ZFC that for every $f:\mathbf{R} \to \mathbf{R}$, there is an $X \subseteq \mathbf{R}$ such that $f|X$ is continuous and X is of second category (or non-Lebesgue measurable)? Clearly, Martin's Axiom would have to fail in any model of ZFC giving an affirmative answer to this question. As pointed out by one of the referees, asking that D be Lebesgue measurable of positive measure is asking too much, for such a set would have to contain a closed set of positive measure, which would be uncountable and therefore of cardinality c, making the Sierpinski-Zygmund function a counterexample.

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