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On category bases: Abstract

In an effort to unify properties which are common for measure and category, John C. Morgan II defined the concept of category base. Namely, $\mathcal{C}$ is said to be a *category base* on X if $\mathcal{C}$ is a family of nonempty subsets of X, called *regions*, satisfying the following axioms: (1) $\cup\mathcal{C} = X$, (2) if $\mathcal{R} \subset \mathcal{C}$ and $A \in \mathcal{C}$ are such that $|\mathcal{R}| < |\mathcal{C}|$, $\mathcal{R}$ is disjoint and $A \cap B$ contains no region for each $B \in \mathcal{R}$, then there exists a region D such that $D \subset A \setminus (\cup\mathcal{R})$. For a survey of results from the theory of category bases the readeris refered to Morgan's book [2]. With respect to a given category base $\mathcal{C}$ on X, one can classify the subsets of X in the following way. A set $E \subset X$ is *singular* if each region contains a subregion disjoint from E. A set $M \subset X$ is *meager* if it is a union of countably many singular sets; the family of all meager sets contains a subregion A such that either $A \cap S$ or $A \setminus S$ is meager; the family of all Baire sets for $\mathcal{C}$ is denoted by $B(\mathcal{C})$. Two category bases $\mathcal{C}_1$ and $\mathcal{C}_2$ on X are *equivalent* if $M(\mathcal{C}_1) = M(\mathcal{C}_2)$ and $B(\mathcal{C}_1) = B(\mathcal{C}_2)$.

A standard example of a category base is the family of all sets of positive measure with respect to a σ-finite complete measure. In this case the meager sets coincide with the sets of measure zero and the Baire sets coincide with the measurable ones. In general, we have the following characterization, which is a partial soultion to Morgan's Measurability Problem.

Theorem 1 *(See [3].) Let $(X, \mathcal{M}, \mu)$ be a complete measure space with the finite subset property and let $\mathcal{N} = \{F \subset X : \mu(F) = 0\}$. There exists a category base $\mathcal{C}$ on X such that $M(\mathcal{C}) = \mathcal{N}$ and $B(\mathcal{C}) = \mathcal{N}$ if and only if the measure μ is decomposable.*

Let us recall that a measure μ is *decomposable* if there exists a disjoint family $\mathcal{R} \subset \mathcal{M}$ of sets of nonzero finite measure such that (i) $X \setminus (\cup\mathcal{R})$ is a zero set, (ii) if $\mu(A) > 0$, then there exists $B \in \mathcal{R}$ such that $\mu(A \cup B) > 0$, (iii) if $Y \subset X$ and $Y \cap B$ is measurable for each $B \in \mathcal{R}$, then Y is measurable.

Another standard example of a category base is any topology (without the expty set). In this case the meager sets coincide with the sets of first category and the Baire sets coincide with the sets with the Baire property.

Theorem 2 *(See[1].) Let $\mathcal{C}$ be a category base on X such that X is not meager. Then $\mathcal{C}$ is equivalent to a topology on X if and only if there exists a lower density function on the σ-field $R(\mathcal{C})$ with respect to the ideal $M(\mathcal{C})$.*

Let us recall that a function f from a field K into K is called a *lower density with respect to a ideal* $I \subset K$ if for any pair A, B in K one has: (i) $f(A)\Delta A$, (ii) $A\Delta B$ implies $f(A) = f(B)$, (iii) $f(\emptyset) = \emptyset$, (iv) $f(A \cap B) = f(A) \cap f(B)$.

For a given category base $\mathcal{C}$ on X, we say that $\mathcal{R}$ is a *category decomposition* of X if $\mathcal{R}$ is a disjoint family of regions such that each region intersects a member of $\mathcal{R}$ in a non–meager set.

Theorem 3 *(See [1].) Let $\mathcal{C}$ be a category base on X such that each region contains a subregion with property P. Then there exists a category decomposition of X consisting exclusively of regions with property P.*

Let $\mathcal{C}$ be a category base on X. Following Morgan [2] we say that a set $E \subset X$ is an *essential hull for a set S* if the following conditions are satisfied: (i) E is a Baire set, (ii) $S \setminus E$ is meager, (iii) if F is a Baire set and $S \setminus F$ is meager, then $E \setminus F$ is meager.

Theorem 4 *(See [1].) If $\mathcal{C}$ is a category base of X, then there exists an essential hull for any subset of X.*

From Theorems 3 and 4 the next theorem follows easily.

Theorem 5 *(J. Morgan and K. Schilling) $R(\mathcal{C})$ is closed under operation (A).*

References

[1] M. Detlefsen and A. Szymanski, *Category bases*, preprint

[2] J. C. Morgan II, *Point set Theory, Pure and Applied Mathematics*, Marcel Dekker, Inc., New Your–Basel, 1990

[3] A. Szymanski, *On the measurability problem for category bases*, preprint.