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CONVERGENCE THEOREMS FOR THE HENSTOCK INTEGRAL

We assume that the reader is familiar with the definitions of the Denjoy-Perron and Henstock integrals. For a collection of tagged intervals $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq N\}$, we will write

$$f(\mathcal{P}) = \sum_{i=1}^{N} f(x_i)(d_i - c_i) \quad \text{and} \quad F(\mathcal{P}) = \sum_{i=1}^{N} \big(F(d_i) - F(c_i)\big).$$

The most general convergence theorem for the Lebesgue integral is the Vitali convergence theorem. It includes both the monotone convergence and dominated convergence theorems as special cases.

VITALI CONVERGENCE THEOREM: Let $\{f_n\}$ be a sequence of Lebesgue integrable functions defined on $[a, b]$, let $F_n(x) = \int_a^x f_n$ for each n, and suppose that $\{f_n\}$ converges to f almost everywhere on $[a, b]$. If the sequence $\{F_n\}$ is equi AC on $[a, b]$, then f is Lebesgue integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Two convergence theorems for the Henstock (Denjoy-Perron) integral have been known for some time. They each contain the monotone convergence and dominated convergence theorems as special cases. We state the definitions for the crucial hypotheses.

DEFINITION: A sequence $\{F_n\}$ of ACG_* functions is equi-uniformly ACG_* on $[a, b]$ if $[a, b] = \bigcup_n E_n$ where the sequence $\{F_n\}$ is equi AC_* on each E_n.

DZVARSEISVILI CONVERGENCE THEOREM: Let $\{f_n\}$ be a sequence of Denjoy-Perron integrable functions defined on $[a, b]$, let $F_n(x) = \int_a^x f_n$ for each n, and suppose that $\{f_n\}$ converges to f almost everywhere on $[a, b]$. If $\{F_n\}$ is equicontinuous and equi-uniformly ACG_* on $[a, b]$, then f is Denjoy-Perron integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

DEFINITION: A sequence $\{f_n\}$ of Henstock integrable functions is uniformly Henstock integrable on $[a, b]$ if for each $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $|f_n(\mathcal{P}) - \int_a^b f_n| < \epsilon$ for all n whenever $\mathcal{P}$ is subordinate to δ on $[a, b]$.

TRIVIAL CONVERGENCE THEOREM: Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a, b]$ and suppose that $\{f_n\}$ converges pointwise to f on $[a, b]$. If the sequence $\{f_n\}$ is uniformly Henstock integrable on $[a, b]$, then f is Henstock integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

These hypotheses appear to be incompatible – neither implies the other. We seek a new convergence theorem that includes both of them.

DEFINITION: Let $\{F_n\}$ be a sequence of functions defined on $[a,b]$ and let $E \subset [a,b]$. The sequence $\{F_n\}$ is $\mathcal{P}$–Cauchy on E if $\{F_n\}$ converges pointwise on E and if for each $\epsilon > 0$ there exist a positive function δ on E and a positive integer N such that $|F_n(\mathcal{P}) - F_m(\mathcal{P})| < \epsilon$ for all $m, n \geq N$ whenever $\mathcal{P}$ is subordinate to δ and all of the tags of $\mathcal{P}$ are in E. The sequence $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on E if E can be written as a countable union of sets on each of which $\{F_n\}$ is $\mathcal{P}$–Cauchy.

It is easy to verify that a $\mathcal{P}$–Cauchy sequence of AC functions is equi AC. Hence the condition $\{F_n\}$ is equi AC in the Vitali convergence theorem can be replaced with $\{F_n\}$ is $\mathcal{P}$–Cauchy. Thus we are led to the following theorem.

GENERAL CONVERGENCE THEOREM: Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$, let $F_n(x) = \int_a^x f_n$ for each n, and suppose that $\{f_n\}$ converges to f almost everywhere on $[a,b]$. If the sequence $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$, then f is Henstock integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

The proof is not too difficult and will appear in the Journal of the London Mathematical Society. There it is also shown that the Dzvarseisvili and trivial convergence theorems are special cases of this result. By modifying the definition of a $\mathcal{P}$–Cauchy sequence, one can extend this convergence theorem to other Riemann-type integrals.