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Transfinite Induction and Integrals

I shall define an integral in the m-dimensional Euclidean space by Marik's idea, and I shall show that this integral can be obtained by extending a variational integral with the standard transfinite induction. The Gauss-Green theorem holds for vector fields which are differentiable outside exceptional sets. The variational integral and its extensions are invariant with respect to continuously differentiable changes of coordinates, and thus they are suitable for integration on differentiable manifolds.

By H we denote the $(m-1)$-dimensional Hausdorff measure in R^m. A compact set $T \subset R^m$ with $H(T) < \infty$ is called thin. A bounded set $A \subset R^m$ is called admissible if $A^{\cdot}$ is thin. We denote by $\mathcal{A}$ the family of all admissible subsets, and $\mathcal{A}(A) = \{B \in \mathcal{A}: B \subset A\}$. We say that a sequence $\{A_n\}$ in $\mathcal{A}$ converges to a set $A \in \mathcal{A}$, and write $\{A_n\} \to A$, if $A_n \subset A$ for $n=1,2,\ldots$, $\sup_n \|A_n\| < \infty$, and $\lim |A - A_n| = 0$. By using the convergence, we can define the closed subfamilites of $\mathcal{A}$, and if $\mathcal{C} \subset \mathcal{A}$ we define the closure $cl\mathcal{C}$ to be the intersections of all the closed subfamilies which contain $\mathcal{C}$. A function $F: \mathcal{A}(A) \to R$ is said to be continuous if $F(B_n) \to F(B)$ for each $\{B_n\} \subset \mathcal{A}(A)$ and $B \in \mathcal{A}(A)$ with $\{B_n\} \to B$.

1 DEFINITION. Let $A \in \mathcal{A}$ and $f: A \to R$. We say that f is v-integrable in A if there is an additive continuous function F on $\mathcal{A}(A)$ which satisfies the following conditions: For each $\varepsilon > 0$ there is a thin set $T \subset A^-$ and a positive function δ such that

$$\Sigma_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each δ-fine ε-partition $\{(A_i, x_i)\}_{i=1}^{p}$ in $A - T$. We denote by $\mathcal{V}(A, f)$ the family of all $B \in \mathcal{A}(A)$ on which f is v-intergrable, and the v-integral of f over a set A in $\mathcal{A}$ is denoted by $I_v(f, A)$.

We set $I_0 = \mathcal{V}(A, f)$ and for each $B \in I_0$, set $F_0(B) = I_v(f, B)$. By the standard transfinite induction, we define I_{ω_1}, and F_{ω_1}, where ω_1 is the first uncountable ordinal. The following definition is due to J. Marik.

2 DEFINITION. Let $A \in \mathcal{A}$, $f: A \to R$. We say that f is integrable in A if $A \in \mathrm{cl}\mathcal{V}(A, f)$, and there is a continuous function F on $\mathcal{A}(A)$ such that $F(B) = I_v(f, B)$, for all $B \in \mathcal{V}(A, f)$. We denote by $\mathfrak{I}(A, f)$ the family of all $B \in \mathcal{A}(A)$ on which f is integrable and the integral of f over a set $A \in \mathcal{A}$ is denoted by $I(f, A)$.

3 PROPOSITION. If $A \in \mathcal{A}$, and $f: A \to R$, then $I_{\omega_1} = \mathfrak{I}(A, f)$ and $I(f, B) = F_{\omega_1}(B)$ for all $B \in \mathfrak{I}(A, f)$.

4 THEOREM. Let $A \in \mathcal{A}$, and let $T \subset A^-$ be a thin set. Let v be a continuous vector field in A^- which is differentiable in $A^0 - T$. Then div v is integrable in A and

$$I(\mathrm{div}\ v, A) = \int_{A^{\cdot}} v \cdot n_A \, dH.$$

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