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Non–Uniformization Results for the Projective Hierarchy

A well known theorem of Kondo–Novikov asserts that any $\underset{\sim}{\Pi}^1_1$ subset P of the plane (which we may take to be $\omega^\omega \times \omega^\omega$) can be uniformized by a $\underset{\sim}{\Pi}^1_1$ relation P'. That is, $\forall x\, [\, \exists y\, P(x,y) \leftrightarrow \exists y\, P'(x,y) \leftrightarrow \exists ! y\, P'(x,y)\,]$. It is also well known that this uniformization property fails for the collection of analytic sets. On the other hand, Lusin and Novikov respectively showed that any $\underset{\sim}{\Sigma}^1_1$ (resp. $\underset{\sim}{\Delta}^1_1$) subset of the plane with all vertical sections countable could be written as a countable union of $\underset{\sim}{\Sigma}^1_1$ $(\underset{\sim}{\Delta}^1_1)$ partial graphs. The question we address here is whether this "dual uniformization" property holds for $\underset{\sim}{\Pi}^1_1$ (it is easy to see that any $\underset{\sim}{\Pi}^1_1$ set with countable sections is the union of ω_1 many $\underset{\sim}{\Pi}^1_1$ graphs).

We first show, working in ZF set theory alone, that the answer is no, in a strong way. Specifically, we have:

THEOREM: There is a $\underset{\sim}{\Pi}^1_1$ set with countable sections which can not be written as the countable union of graphs each of which is in the σ algebra generated by the $\underset{\sim}{\Sigma}^1_2$ sets.

Assuming stronger axioms of set theory, we then determine exactly where one picks up such a countable covering. In fact, we do this for all levels $\underset{\sim}{\Pi}^1_{2n+1}$ of the projective hierarchy. We have:

THEOREM: Assume $\bigcup_k \Game^{2n}\, \omega\cdot k\text{–}\underset{\sim}{\Pi}^1_1$ determinacy. There is a $\underset{\sim}{\Pi}^1_{2n+1}$ set $P \subset \omega^\omega \times \omega^\omega$ with all sections countable and such that for all $k \in \omega$, P can not be written as a countable union of (partial) graphs $P = \bigcup_m G_m$ with each G_m in the class $\Game^{2n+1}\, \omega\cdot k\text{–}\underset{\sim}{\Pi}^1_1$. However, every $P \in \underset{\sim}{\Sigma}^1_{2n+2}$ with countable sections can be written as countable union of (partial) graphs, each in the class $\bigcup_k \Game^{2n+1}\, \omega\cdot k\text{–}\underset{\sim}{\Pi}^1_1$.

We observe also some miscellaneous facts about such countable uniformizations. Finally, we ask how far our first theorem can be extended in ZF alone.

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