T.I. Ramsamujh Florida International University, Miami, Florida

THE FAMILY OF COMPACT POROUS SETS

Let K(R) be the Polish space of all non-empty compact subsets of R, and PS be the family of all porous sets in K(R). Then it is easy to show that

Proposition 1: PS is a coanalytic subset of K(R).

A natural question is whether PS is a Borel subset of K(R). (A negative answer would rule out any Borel characterisation of the set PS.) To answer this question we introduce a natural rank function on PS. (All the concepts that we use are standard and can be found in [1], [2], or [3].)

For each $M > 0$ and $A \in K(R)$, let T(M,A) be the tree defined by: $\langle I_1, \ldots, I_n \rangle \in T(M,A)$ if and only if

(i) $I_{i+1} \subseteq I_i$, $|I_i| \le 1/i$, endpoints$(I_i) \in Q$, and

(ii) for all J with $I_n \subseteq J \subseteq I_i$, $\lambda[A;J] \le M.|J|/i$.

Here the I_i's and J denote closed intervals and $\lambda[A;J]$ is the length of the longest subinterval of J which contains no points of A. We then have

Proposition 2: $A \in PS \iff$ for all $M > 0$, T(M,A) is well-founded.

Proposition 3: For $A \in PS$, $\sup\{|T(M,A)|+1 : M > 0\}$ is always a limit ordinal.

This allows us to define the rank $|A|$ of A as the unique ordinal α such that

$$\sup\{|T(M,A)|+1 : M > 0\} = \omega.\alpha.$$

The rank function provides a natural measure of the complexity of the sets in PS and we actually have

Proposition 4: $|A|=1$ if and only if A is uniformly porous.

Proposition 5: $|.| : PS \to \omega_1$ is a coanalytic norm.

Proposition 6: For each $\alpha < \omega_1$, there is a set $A \in PS$ with $|A|=\alpha$.

By the Boundedness Principle we then get

Corollary 7: PS is not a Borel subset of K(R).

REFERENCES

[1] A.S. Kechris & W.H. Woodin, Ranks of differentiable functions, Mathematica 33 (1986), 252-278.

[2] Y.N. Moschovakis, Descriptive set theory (North Holland, New York 1980).

[3] T.I. Ramsamujh, The complexity of nowhere differentiable continuous functions, Canadian J. Math 41 (1989), 83-105.