Fractals determined by the Weierstrass functions

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The term "fractal" was coined by Benoit Mandelboot for sets E whose "dimension" is greater than the topological dimension. There are many apparently different ways of defining dimension, but I have suggested (see [7]) that we should reserve the name fractal for those sets which are sufficiently regular to ensure all those definitions produce the same answer. This is a mild regularity condtion which is usually satisfied (when it can be checked) by examples related to the physical world. Our object is to reformulate some old problems related to the class of functions defined by Weierstrass each of which is continuous but nowhere differentiable. We believe that a solution to some or all of these problems should now be accessible. For a recent summary of analytic results, the reader is referred to [3]. The paper [7] surveys the definitions of fractal measures, including packing measure defined first in [6], so we do not repeat those.

Suppose $0<\alpha<1$, $b>1$ and $\Phi: \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period 1. Define

$$f(t) = f_{\alpha,b}(t) = \sum_{n=0}^{\infty} b^{-\alpha n}\Phi(b^n t) \tag{1}$$

and consider the planar set $G = \{(x,f(x)): x \in [0,1]\}$ and the linear sets $E_c = \{x \in [0,1]: f(x) = c\}$. Weierstrass considered $\Phi(x) = \cos 2\pi x$, and van der Waerden-Tagaki considered $\Phi(x) = \text{dist}\ (x,\mathbb{Z})$.

It is not too difficult to prove from known results that the packing dimension of G satisfies

$$\text{Dim } G = 2-\alpha$$

so G is a fractal if and only if the Hansdorff dimension satisfies

$$\dim G = 2-\alpha .$$

This is known to be the case whenever f satisfies a strong self-simularity condition; for example, if $f(x) = \Phi_1(x)-\Phi_2(x)$ where $\Phi_1(x)$, $\Phi_2(x)$ are the coordinates of a Peano space filling curve, Kono [4] proves that

$$0<s^{3/2}\text{-}m(G)<s^{3/2}\text{-}p(G)<x\infty \tag{2}$$

so G is a fractal of index 3/2.

Let me now state the main conjecture.

CONJECTURE 1. *For a wide class of Φ, including the Weierstrass* *case,* (1) *defines a continuous* f *whose graph* G *is a fractal of index* $2-\alpha$.

In order to prove the conjecture for a particular Φ, it is sufficient to show that the Hansdorff dimension satisfies

$$\dim G \geq 2-\alpha. \tag{3}$$

Whenever $\alpha = 1$, the result corresponding to (3) is trivial, so we do know that $\dim G = \text{Dim } G = 1$. However, this G is not a fractal since it also has topological dimension 1. For some recent progress in the analysis of this critical case see Anderson and Pitt [1]. Recently, Mauldin and Williams [5], showed that $\dim G \geq 2-\alpha-c/\log b$ whenever b is large enough. As pointed out by Kono [4], (3) would also follow for the Weierstrass functions if we could show that f has a continuous occupation density. For a survey of the connections between properties of the occupation density and those of the function f see [2]. There is every reason to believe this is the case, but no one seems to have proved it.

If conjecture 1 is true one could ask whether there are exact growth functions which give G finite positive Hansdorff measure or packing measure. In particular, is the analogue of (2) true with 3/2 replaced by $(2-\alpha)\lambda$?

CONJECTURE 2 *For a wide class of* Φ, *including the Weierstrass functions,* (1) *defines a funciton* f *whose level sets or either empty or are fractals of index* $1-\alpha$.

Meaningful but partial information exists for the critical case $\alpha = 1$. General section theorems would yield information about most level sets if conjecture 1 were established.

It is well know that, if f is replaced by a standard Brownian motion, then the graph is a fractal of index 3/2 a.s., and the non-empty level sets are fractals of index $\frac{1}{2}$. In fact, exact Hansdorff measure functions are known for both G and E_c see [7]. It is less well known that the introduction of random factors in the definition (1) allows us to prove the behaviour which we conjecture for the deterministic case.

THEOREM *Suppose* $\{X_n\}$ *is a sequence of independent* N(0,1) *random variables and*

$$f(t,w) = \sum_{n=0}^{\infty} b^{-\alpha n}\Phi(b^n t)X_n$$

where $\Phi: \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous, periodic and not constant. Then, with probability* 1, *the graph* G *of* f(t,w) *satisfies*

$$\dim G = \mathrm{Dim}\, G = 2-\alpha$$

S. James Taylor
Department of Mathematics
University of Virginia

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