Possible Generalizations of Plessner's Theorem

by

J. Marshall Ash, DePaul University, Chicago, IL 60614

Let $f(x,y) = \Sigma\, c_{mn} e^{i(mx+ny)}$ be a two dimensional trigonometric series, and let $f^j(x,y) = \Sigma\, M^j_{mn} c_{mn} e^{i(mx+ny)}$ be a "conjugate" trigonometric series. Of particular interest are the four choices $M^1_{mn} = \frac{m}{(m^2+n^2)^{.5}}$, $M^2_{mn} = \frac{mn}{m^2+n^2}$, $M^3_{mn} = \text{sgn } m$, and $M^4_{mn} = \text{sgn } mn$. Also consider the following modes of convergence: 1 = square, 2 = restricted rectangular, 3 = unrestricted rectangular, 4 = circular, and 5 = triangular. Let E be any subset of $[0,2\pi]x[0,2\pi]$ of positive Lebesgue measure. We then form 100 statements. (All are putative generalizations of Plessner's basic result for one dimensional trigonometric series. See page 216 of [7].)

Statement (i,j,k). If f converges in mode i at each point of E, then f^j converges in mode k at almost every point of E.

I will not consider triangular convergence here, thereby reducing our fields of inquiry to 64 cases. When $i < k \leq 3$ and $j \in \{3, 4\}$, Statements (i,j,k) are trivially false, since for example square convergence of a series does not force a. e. restriced rectangular convergence of that series. (See [4].) This eliminates 6 cases. It is also very unlikely and probably not difficult to prove that Statements (i,j,k) are false when $i < k \leq 3$ and $j \in \{1, 2\}$. This leaves 52 potential theorem.

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It has been shown that Statements (3,j,1), $j \in \{3, 4\}$ are false.[1], [2], [5] Consequently, Statements (i,j,k) are false for $(i,j,k) \in \{1, 2, 3\} \times \{3, 4\} \times \{1, 2, 3\}$. Also Statement (4,3,4) is false.[2] In fact, it is probable that all of the 26 remaining statements involving $j \in \{3, 4\}$ are false.

We now pass to the 26 remaining cases associated with $j \in \{3, 4\}$. Even here I do not expect any connection between circular and the other modes of convergence to hold. This would remove 12 cases. Statements (3,1,k), k = 1, 2, and Statements (3,2,k), k = 1, 2, 3 are all <u>true</u>.[6], [3]

We are left with 9 substantial questions. They are Statements (4,j,4), $j \in \{1, 2\}$, Statement (3,1,3), and Statements (i,j,k), $2 \geq i \geq k$, $j \in \{1, 2\}$. The first three of these are probably the most interesting, so we will close by restating them without the messy notation.

Question 1. If f converges circularly on E, does f^1 converge circularly a. e. on E? (See Statement (4,1,4).)

Question 2. If f converges circularly on E, does f^2 converge circularly a. e. on E? (See Statement (4,2,4).)

Question 3. If f converges unrestrictedly rectangularly on E, does f^1 converge unrestrictedly rectangularly on E? (See Statement (3,1,3).)

References

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