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EQUIVALENCE OF THE GENERALIZED RIEMANN AND RESTRICTED DENJOY INTEGRALS[1]

It has been known for some time that the restricted Denjoy integral (D_* integral) and the generalized Riemann integral (GR integral) are equivalent. That is, a function is D_* integrable if and only if it is GR integrable and the integrals are equal. The early proofs of this fact ([1] and [3]) proceeded by proving that the GR integral is equivalent to the Perron integral. The result then follows since the Perron integral is equivalent to the D_* integral [8]. In the last few years two papers ([4] and [9]) have appeared that offer direct proofs. However, both of these papers have an error in the proof that the indefinite GR integral is ACG_*.

The purpose of this paper is to present a direct proof of the equivalence of the D_* and GR integrals. For completeness and consistency of notation the definitions of the integrals and proofs of all the main results are included. We feel that this is necessary since the proofs are scattered throughout the literature and some of the proofs are lacking in detail.

We begin with the definitions of the integrals. In order to define the D_* integral the concepts of bounded variation and absolute continuity must be extended. The necessary extensions appear in the definition below. If f is defined on [c,d], then $\omega(f,[c,d]) = \sup \{|f(t) - f(s)| : c \leq s < t \leq d\}$ is the oscillation of f on [c,d].

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[1]This material is to be included in the author's doctoral dissertation under preparation at the University of Illinois under the direction of J. J. Uhl, Jr.

DEFINITION 1: Let $f : [a,b] \longrightarrow \mathbb{R}$ and let $E \subset [a,b]$.

(a) The function f is BV_* on E if $\sup \{\sum_i \omega(f,[a_i,b_i])\}$ is finite where the supremum is taken over all finite collections of non-overlapping intervals that have endpoints in E.

(b) The function f is BVG_* on E if E can be expressed as a countable union of sets on each of which f is BV_*.

(c) The function f is AC_* on E if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \omega(f,[a_i,b_i]) < \varepsilon$ whenever $\{[a_i,b_i]\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_i (b_i - a_i) < \delta$.

(d) The function f is ACG_* on E if f is continuous on E and if E can be expressed as a countable union of sets on each of which f is AC_*.

(e) The function f is AC on E if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i |f(b_i) - f(a_i)| < \varepsilon$ whenever $\{[a_i,b_i]\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_i (b_i - a_i) < \delta$.

The basic properties of these functions can be found in Saks [8]. We will require the following two theorems. The first is a slight modification of Saks' Theorem 9.1 on page 233 of [8]. Its proof is quite similar to the proof given by Saks. The second is Saks' Theorem 8.8 also on page 233 of [8]. Recall that a portion of a set E is a nonempty set P of the form $P = E \cap I$ where I is an open interval.

THEOREM 2: Let E be a closed set in $[a,b]$ and let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on E. Then f is BVG_* (ACG_*) on E if and only if every perfect subset of E contains a portion on which f is $BV_*(AC_*)$.

THEOREM 3: Let E be a closed set with bounds a and b and let $f : [a,b] \longrightarrow \mathbb{R}$. Then f is AC_* on E if and only if f is both AC and BV_* on E.

It is well-known that an ACG_* function is differentiable almost everywhere [8, p. 230]. Thus, the definition of the D_* integral is a direct extension of the descriptive definition of the Lebesgue integral.

DEFINITION 4: The function $f : [a,b] \longrightarrow \mathbb{R}$ is D_* integrable on $[a,b]$ if there exists a function $F : [a,b] \longrightarrow \mathbb{R}$ such that F is ACG_* on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$.

The definition of the generalized Riemann integral extends the Riemann integral by using variable mesh size as opposed to insisting on a uniform mesh ([2] and [4]). This seemingly minor modification has far reaching consequences. Before defining the GR integral we develop some notation and terminology.

DEFINITION 5: Let $\delta(\cdot)$ be a positive function defined on the interval $[a,b]$. A tagged interval $(s,[c,d])$ consists of an interval $[c,d] \subset [a,b]$ and a point $s \in [c,d]$. The tagged interval $(s,[c,d])$ is subordinate to δ if $[c,d] \subset (s - \delta(s), s + \delta(s))$. Script capital letters such as $\mathcal{P}$ and $\mathcal{D}$

will be used to denote finite collections of non-overlapping tagged intervals. Let $\mathcal{P} = \{(s_i,[c_i,d_i]) : 1 \le i \le N\}$ be such a collection in $[a,b]$.

(a) The points $\{s_i : 1 \le i \le N\}$ are called the tags of $\mathcal{P}$.

(b) The intervals $\{[c_i,d_i] : 1 \le i \le N\}$ are called the intervals of $\mathcal{P}$.

(c) If $(s_i,[c_i,d_i])$ is subordinate to δ for each i, then we write $\mathcal{P}$ is sub δ.

(d) If $[a,b] = \bigcup_{i=1}^{N} [c_i,d_i]$, then $\mathcal{P}$ is called a tagged partition of $[a,b]$.

(e) If $\mathcal{P}$ is a tagged partition of $[a,b]$ and if $\mathcal{P}$ is sub δ, then we write $\mathcal{P}$ is sub δ on $[a,b]$.

(f) If $f : [a,b] \longrightarrow \mathbb{R}$, then $f(\mathcal{P}) = \sum_{i=1}^{N} f(s_i)(d_i - c_i)$.

(g) If F is defined on the intervals of $[a,b]$, then

$$F(\mathcal{P}) = \sum_{i=1}^{N} F([c_i,d_i]).$$

DEFINITION 6: The function $f : [a,b] \longrightarrow \mathbb{R}$ is GR integrable on $[a,b]$ if there exists a real number α with the following property: for each $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - \alpha| < \varepsilon$ whenever $\mathcal{P}$ is sub δ on $[a,b]$. The function f is GR integrable on $E \subset [a,b]$ if $f\chi_E$ is GR integrable on $[a,b]$ where χ_E is the characteristic function of E.

The GR integral has the usual properties of an integral; uniqueness, integrability on subintervals, and linearity. The proofs of these facts can be found in McLeod [5].

Let f be GR integrable on $[a,b]$ and let $F(t) = \int_a^t f$; that is, F is the indefinite GR integral of f. We adopt the convention that F is an interval function when defined on tagged intervals. Thus, $F(s,[c,d]) = F(d) - F(c) = \int_c^d f$.

Let f and F be as in Definition 5 and let δ be a positive function on $[a,b]$. Then $(s,[c,d])$ is sub δ if and only if $\{(s,[c,s]), (s,[s,d])\}$ is sub δ and we have

$$f(s)(d - c) = f(s)(s - c) + f(s)(d - s)$$

$$F(d) - F(c) = F(s) - F(c) + F(d) - F(s).$$

Therefore, if $\mathcal{P}$ is sub δ on $[a,b]$ the values of $f(\mathcal{P})$ and $F(\mathcal{P})$ remain unchanged if we assume either (1) or (2) below.

(1) All of the tags of $\mathcal{P}$ occur as endpoints.

(2) Each tag of $\mathcal{P}$ occurs only once.

It will sometimes be convenient to make one of these assumptions.

The next result, which is often referred to as Henstock's Lemma, plays a crucial role in the theory of the GR integral. The proof is not difficult and can be found in McLeod [5].

LEMMA 7 (Henstock's Lemma): Let $f : [a,b] \longrightarrow \mathbb{R}$ be GR integrable on $[a,b]$ and let $F(t) = \int_a^t f$. Given $\varepsilon > 0$ choose a positive function δ on $[a,b]$ so that $|f(\mathcal{P}) - F(b)| < \varepsilon$ whenever $\mathcal{P}$ is sub δ on $[a,b]$. If $\mathcal{D} = \{(s_i,[c_i,d_i]) : 1 \leq i \leq N\}$ is sub δ, then

$$|f(\mathcal{D}) - F(\mathcal{D})| \le \varepsilon \quad \text{and} \quad \sum_{i=1}^{N} |f(s_i)(d_i - c_i) - (F(d_i) - F(c_i))| \le 2\varepsilon.$$

We first prove that every D_* integrable function is GR integrable. The idea for the proof is due to Yee and Naak-in [9]. It is a creative proof, but some of the details are missing and the function δ is not clearly defined. The proof below eliminates these difficulties.

THEOREM 8: If $f : [a,b] \longrightarrow \mathbb{R}$ is D_* integrable on $[a,b]$, then f is GR integrable on $[a,b]$ and the integrals are equal.

Proof: Let $F(t) = (D_*)\int_a^t f$. Then F is ACG_* on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$. Let $\{E_i\}$ be a collection of pairwise disjoint sets such that $[a,b] = \bigcup_i E_i$ and F is AC_* on each E_i. Let H be the set of measure zero such that $F' = f$ on $[a,b] - H$ and let $H = \bigcup_{n=1}^{\infty} H_n$ where $H_1 = \{t \in H : |f(t)| \le 1\}$ and $H_n = \{t \in H : 2^{n-2} < |f(t)| \le 2^{n-1}\}$ for each $n \ge 2$. For each pair of positive integers i and n write $E_i \cap H_n = G_{in} \cup \{t_{in}^k\}$ where G_{in} consists of all the points of $E_i \cap H_n$ that are limit points of $E_i \cap H_n$ on both sides. The remaining points of $E_i \cap H_n$ form a countable set. (See [8], p. 260).

Let $\varepsilon_1 > 0$ and let $\varepsilon = \varepsilon_1(b - a + 5)^{-1}$. Choose $\delta_{in} > 0$ so that $\sum_j \omega(F,[\alpha_j,\beta_j]) < \varepsilon 2^{-n-i}$ whenever $\{[\alpha_j,\beta_j]\}$ is a finite collection of non-overlapping intervals that have endpoints in $E_i \cap H_n$ and satisfy $\sum_j (\beta_j - \alpha_j) < \delta_{in}$. Let $\mathcal{O}_{in}$ be an open set containing $E_i \cap H_n$ such that

$\mu(\mathcal{O}_{in}) < \min\{\delta_{in}, \varepsilon 2^{-2n-i}\}$. For each $t \in G_{in}$ choose c_{in}^t and d_{in}^t in $E_i \cap H_n$ such that $c_{in}^t < t < d_{in}^t$ and $[c_{in}^t, d_{in}^t] \subset \mathcal{O}_{in}$. Use the continuity of F to choose $\eta_{in}^k > 0$ so that $|F(s) - F(t_{in}^k)| < \varepsilon 2^{-n-i-k}$ whenever $|s - t_{in}^k| < \eta_{in}^k$ and $s \in [a,b]$.

Now we define δ on $[a,b]$.

(i) If $t \in [a,b] - H$, then choose $\delta(t) > 0$ so that

$$|F(s) - F(t) - f(t)(s - t)| < \varepsilon|s - t|$$

whenever $s \in (t - \delta(t), t + \delta(t)) \cap [a,b]$.

(ii) If $t \in G_{in}$, then $\delta(t) = \min\{t - c_{in}^t, d_{in}^t - t\}$.

(iii) If $t = t_{in}^k$, then choose $\delta(t) > 0$ so that $\delta(t) < \eta_{in}^k$ and $(t - \delta(t), t + \delta(t)) \subset \mathcal{O}_{in}$.

This defines δ on all of $[a,b]$.

We must show that $|f(\mathcal{P}) - F(b)| < \varepsilon_1$ whenever $\mathcal{P}$ is sub δ on $[a,b]$. Let $\mathcal{P}$ be sub δ on $[a,b]$ and assume without loss of generality that each tag occurs only once. Let $\mathcal{P}_d$ be the subset of $\mathcal{P}$ that has tags in $[a,b] - H$ and let $\mathcal{P}_{in}$ be the subset of $\mathcal{P}$ that has tags in $E_i \cap H_n$. Let π be the set of pairs (i,n) for which $\mathcal{P}_{in} \neq \emptyset$. Then

$$|f(\mathcal{P}) - F(b)| = |f(\mathcal{P}_d) + \sum_\pi f(\mathcal{P}_{in}) - F(\mathcal{P}_d) - \sum_\pi F(\mathcal{P}_{in})|$$

$$\leq |f(\mathcal{P}_d) - F(\mathcal{P}_d)| + \sum_\pi |f(\mathcal{P}_{in})| + \sum_\pi |F(\mathcal{P}_{in})|.$$

If $(s,[u,v]) \in \mathcal{P}_d$, then

$$|F(v) - F(u) - f(s)(v - u)| \leq |F(v) - F(s) - f(s)(v - s)| + |F(s) - F(u) - f(s)(s - u)|$$

$$< \varepsilon(v - s) + \varepsilon(s - u)$$
$$= \varepsilon(v - u).$$

This result, along with the fact that $|f(\mathcal{P}_{in})| \le 2^{n-1}\mu(\mathcal{O}_{in}) < \varepsilon 2^{-n-i}$, implies that

$$(1) \qquad |f(\mathcal{P}) - F(b)| < \varepsilon(b - a) + \varepsilon + \sum_{\pi} |F(\mathcal{P}_{in})|.$$

For each $(i,n) \in \pi$ let $\mathcal{P}'_{in}$ be the subset of $\mathcal{P}_{in}$ that has tags in G_{in} and let $\mathcal{P}^k_{in}$ be the subset of $\mathcal{P}_{in}$ for which the tag is t^k_{in}. Let $\mathcal{P}'_{in} = \{(s_j,[u_j,v_j]) : 1 \le j \le m\}$ and assume without loss of generality that $s_j < s_{j+1}$ for each j. Let $u'_1 = c^{s_1}_{in}$ and for $2 \le j \le m$ let $u'_j = \max\{s_{j-1}, c^{s_j}_{in}\}$. Now $\{[u'_j, s_j] : 1 \le j \le m\}$ is a collection of non-overlapping intervals that have endpoiints in $E_i \cap H_n$. In addition, $\sum_{j=1}^{m} (s_j - u'_j) < \delta_{in}$ since all of the intervals are inside $\mathcal{O}_{in}$. Let $v'_m = d^{s_m}_{in}$ and for $1 \le j \le m - 1$ let $v'_j = \min\{s_{j+1}, d^{s_j}_{in}\}$. Now $\{[s_j, v'_n] : 1 \le j \le m\}$ is also a collection of non-overlapping intervals that have endpoints in $E_i \cap H_n$. In addition, $\sum_{j=1}^{m} (v'_j - s_j) < \delta_{in}$ since all of the intervals are inside $\mathcal{O}_{in}$. We thus have

$$(2) \qquad |F(\mathcal{P}'_{in})| \le \sum_{j=1}^{m} |F(v_j) - F(u_j)|$$

$$\le \sum_{j=1}^{m} |F(v_j) - F(s_j)| + \sum_{j=1}^{m} |F(s_j) - F(u_j)|$$

$$\le \sum_{j=1}^{m} \omega(F,[s_j,v'_j]) + \sum_{j=1}^{m} \omega(F,[u'_j, s_j])$$

$$< 2\varepsilon 2^{-n-i} .$$

Let $\mathcal{P}^k_{in} = \{(t^k_{in}, [u,v])\}$ and compute

$$|F(\mathcal{P}^k_{in})| = |F(v) - F(u)| \tag{3}$$

$$\le |F(v) - F(t^k_{in})| + |F(t^k_{in}) - F(u)|$$

$$< 2\varepsilon 2^{-n-i-k}.$$

Let $\pi_{in} = \{k : \mathcal{P}^k_{in} \ne \emptyset\}$. Then (1), (2), and (3) together imply that

$$|f(\mathcal{P}) - F(b)| < \varepsilon(b - a + 1) + \sum_{\pi} |F(\mathcal{P}'_{in})| + \sum_{\pi}\sum_{\pi_{in}} |F(\mathcal{P}^k_{in})|$$

$$< \varepsilon(b - a + 1) + 2\varepsilon + \sum_{\pi} 2\varepsilon 2^{-n-i}$$

$$< \varepsilon(b - a + 5)$$

$$= \varepsilon_1.$$

Therefore, the function f is GR integrable on $[a,b]$ and $(GR)\int_a^b f = F(b) = (D_*)\int_a^b f.$

Next, we will prove that every GR integrable function is D_* integrable. Let $f : [a,b] \longrightarrow \mathbb{R}$ be GR integrable and let $F(t) = \int_a^t f$. We must show that $F' = f$ almost everywhere on $[a,b]$ and that F is ACG_* on $[a,b]$. We begin by proving that F is continuous.

THEOREM 9: Let $f : [a,b] \longrightarrow \mathbb{R}$ be GR integrable on $[a,b]$. If $F(t) = \int_a^t f$, then F is continuous on $[a,b]$.

Proof: Let $t_0 \in [a,b)$. We will prove that F is continuous on the right at t_0. Let $\varepsilon > 0$ and choose a positive function δ on $[a,b]$ so that $|f(\mathcal{P}) - F(b)| < \varepsilon/3$ whenever $\mathcal{P}$ is sub δ on $[a,b]$. Let $\eta = \min\{\delta(t_0), (1 + |f(t_0)|)^{-1}\varepsilon/3\}$ and fix $\mathcal{D}$ sub δ on $[a,t_0]$. Let $s \in [a,b] \cap [t_0, t_0 + \eta)$ and let $\mathcal{D}' = \mathcal{D} \cup (t_0, [t_0,s])$. Then $\mathcal{D}'$ is sub δ on $[a,s]$ and using Henstock's Lemma we obtain

$$
\begin{aligned}
|F(s) - F(t_0)| &= |F(s) - f(\mathcal{D}') + f(t_0)(s - t_0) + f(\mathcal{D}) - F(t_0)| \\
&< |f(\mathcal{D}') - F(s)| + |f(t_0)|\eta + |f(\mathcal{D}) - F(t_0)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}
$$

Thus, the function F is continuous on the right at t_0.

Similarly, we can prove that F is continuous on the left at each point of $(a,b]$. Hence, the function F is continuous on $[a,b]$.

We turn now to the proof that $F' = f$ almost everywhere. Henstock [1] and Kubota [4] prove this result using the variational integral defined by Henstock [1]. Pu [7] accomplishes the same result but his proof is lengthy and follows an argument similar to the proof for the indefinite Perron integral. In [6] McShane offers an elementary proof but his proof is complicated by the fact that he avoids the use of the Vitali Covering Lemma. (McShane's proof applies to the GR integral even though his definition of integral is not as general as the one used in this paper). The simplest proof that $F' = f$ almost everywhere is found in Yee and Naak-in [9]. The proof has an easily corrected error. In [9] the potential problem is that the elements of their Vitali covering may be too large to guarantee the necessary properties. Our proof follows their argument and avoids this ambiguity.

THEOREM 10: Let $f : [a,b] \longrightarrow \mathbb{R}$ be GR integrable on $[a,b]$. If $F(t) = \int_a^t f$, then $F' = f$ almost everywhere on $[a,b]$.

Proof: Let A^+ be the set of all points t in $[a,b)$ such that either $F^+(t)$ does not exist or $F^+(t) \neq f(t)$ where $F^+(t)$ is the right-hand derivative of F at t. For each $t \in A^+$ there exists $\eta_t > 0$ with the following property: for each $\beta > 0$ there exists $v_\beta^t \in [a,b] \cap (t, t+\beta)$ such that

$$|F(v_\beta^t) - F(t) - f(t)(v_\beta^t - t)| \geq \eta_t(v_\beta^t - t).$$

Let $A_n^+ = \{t \in A^+ : \eta_t \geq \frac{1}{n}\}$. We will show that $\mu^*(A_n^+) = 0$ for each n where $\mu^*(A_n^+)$ denotes the outer Lebesgue measure of A_n^+.

Fix a positive integer n and let $\varepsilon > 0$ be given. Choose a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - F(b)| < \frac{\varepsilon}{4n}$ whenever $\mathcal{P}$ is sub δ on $[a,b]$. The collection of intervals $\mathcal{I} = \{[t, v^t_\beta] : t \in A^+_n, \beta < \delta(t)\}$ forms a Vitali covering of A^+_n. By the Vitali Covering Lemma there exists a finite collection $\{[c_i, d_i] : 1 \leq i \leq N\}$ of disjoint intervals in $\mathcal{I}$ such that $\mu^*(A^+_n) \leq \sum_{i=1}^{N} (d_i - c_i) + \varepsilon/2$. Note that each $(c_i, [c_i, d_i])$ is sub δ and that

$$(d_i - c_i)\eta_{c_i} \leq |F(d_i) - F(c_i) - f(c_i)(d_i - c_i)|.$$

Using Henstock's Lemma we obtain

$$\begin{aligned}
\sum_{i=1}^{N} (d_i - c_i) &\leq \sum_{i=1}^{N} \frac{1}{\eta_{c_i}} |F(d_i) - F(c_i) - f(c_i)(d_i - c_i)| \\
&\leq n \sum_{i=1}^{N} |f(c_i)(d_i - c_i) - (F(d_i) - F(c_i))| \\
&\leq n \cdot \frac{2\varepsilon}{4n} \\
&= \varepsilon/2.
\end{aligned}$$

It follows that $\mu^*(A^+_n) \leq \varepsilon$ and since $\varepsilon > 0$ was arbitrary we conclude that $\mu^*(A^+_n) = 0$.

Since $A^+ = \bigcup_n A_n^+$ we see that $\mu^*(A^+) = 0$. In an analogous manner we can show that $\mu^*(A^-) = 0$ where A^- is the set of all points t in $(a,b]$ such that either $F^-(t)$ does not exist or $F^-(t) \neq f(t)$ where $F^-(t)$ is the left-hand derivative of F at t. Since the set of all points t in $[a,b]$ for which either $F'(t)$ does not exist or $F'(t) \neq f(t)$ is contained in $A^+ \cup A^-$ it follows that $F' = f$ almost everywhere on $[a,b]$.

It remains to be shown that F is ACG_* on $[a,b]$. The technique of the next proof is used in [4] and [9] to prove this fact. This technique will be used in the proof of the next theorem to show that F is BVG_* on $[a,b]$. However, this argument fails to prove that F is ACG_* on $[a,b]$ since the sets E_{jm}^k which appear in the proof depend on the choice of δ and hence on ε. A separate proof that F is ACG_* on $[a,b]$ is required.

THEOREM 11: Let $f : [a,b] \longrightarrow \mathbb{R}$ be GR integrable on $[a,b]$. If $F(t) = \int_a^t f$, then F is BVG_* on $[a,b]$.

Proof: Fix a function δ on $[a,b]$ with $0 < \delta(t) < 1$ for all $t \in [a,b]$ such that $|f(\mathcal{P}) - F(b)| < 1$ whenever $\mathcal{P}$ is sub δ on $[a,b]$. For each pair (j,m) of positive integers define

$$E_{jm} = \{t \in [a,b] : j - 1 \leq |f(t)| < j \text{ and } \tfrac{1}{m+1} \leq \delta(t) < \tfrac{1}{m}\}.$$

It is clear that $[a,b] = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} E_{jm}$. To complete the proof it is sufficient to prove that F is BVG_* on each E_{jm}.

Fix a particular nonempty E_{jm} and choose a positive integer L such that $a + \frac{L}{2m} \leq b < a + \frac{L+1}{2m}$. For each $k = 1, 2, \ldots, L$ let $E^k_{jm} = E_{jm} \cap [a + \frac{k-1}{2m}, a + \frac{k}{2m})$ and let $E^{L+1}_{jm} = E_{jm} \cap [a + \frac{L}{2m}, b]$. Then $E_{jm} = \bigcup_{k=1}^{L+1} E^k_{jm}$ and we claim that F is BV_* on E^k_{jm} for each k.

Let $\{[c_i, d_i] : 1 \leq i \leq N\}$ be a collection of non-overlapping intervals that have endpoints in E^k_{jm}. Since F is continuous we can select points $u_i, v_i \in [c_i, d_i]$ with $u_i < v_i$ such that $|F(v_i) - F(u_i)| = \omega(F, [c_i, d_i])$. Note that the tagged intervals $(c_i, [c_i, u_i])$, $(d_i, [v_i, d_i])$, and $(c_i, [c_i, d_i])$ are subordinate to δ. Using Henstock's Lemma we obtain

$$\sum_{i=1}^{N} \omega(F, [c_i, d_i]) = \sum_{i=1}^{N} |F(u_i) - F(v_i)|$$

$$\leq \sum_{i=1}^{N} \{|F(u_i) - F(c_i) - f(c_i)(u_i - c_i)| + |f(c_i)(u_i - c_i)|$$

$$+ |F(c_i) - F(d_i) + f(c_i)(d_i - c_i)| + |f(c_i)(d_i - c_i)|$$

$$+ |F(d_i) - F(v_i) - f(d_i)(d_i - v_i)| + |f(d_i)(d_i - v_i)|\}$$

$$= \sum_{i=1}^{N} |F(d_i) - F(c_i) - f(c_i)(d_i - c_i)|$$

$$+ \sum_{i=1}^{N} \{|F(u_i) - F(c_i) - f(c_i)(u_i - c_i)|$$

$$+ \ |F(d_i) - F(v_i) - f(d_i)(d_i - v_i)|\}$$

$$+ \sum_{i=1}^{N} \{|f(c_i)|(u_i - c_i) + |f(d_i)|(d_i - v_i)\}$$

$$+ \sum_{i=1}^{N} |f(c_i)|(d_i - c_i)$$

$$\leq 2 + 2 + j \cdot \frac{1}{2m} + j \cdot \frac{1}{2m}$$

$$= 4 + \frac{j}{m} .$$

Therefore, the function F is BV_* on E^k_{jm} and the proof is complete.

In order to give a direct proof that the indefinite GR integral is ACG_* (see Theorem 18) a few preliminary results are needed. The next theorem is interesting in its own right in the theory of the GR integral. It has been proved for the Perron integral ([8], p. 249), the D_* integral ([8], p. 257), and Henstock's variational integral ([1], p. 118 and [4], p. 515). We present what we hope is the first direct proof of this fact for the GR integral.

THEOREM 12: Let E be a bounded, closed set with bounds a and b and let $\{(a_k, b_k)\}$ be the sequence of intervals in $[a,b]$ contiguous to E. Suppose that $f : [a,b] \longrightarrow \mathbb{R}$ is GR integrable on E and on each interval $[a_k, b_k]$. If $\sum_k \omega(\int_{a_k}^{t} f, [a_k, b_k]) < \infty$, then f is GR integrable on $[a,b]$

and $$\int_a^b f = \int_a^b f\chi_E + \sum_k \int_{a_k}^{b_k} f.$$

Proof: Let $g = f - f\chi_E$. We will prove that g is GR integrable on $[a,b]$ and that $\int_a^b g = \sum_k \int_{a_k}^{b_k} f$. Adding $\int_a^b f\chi_E$ to both sides will give the desired result.

Given an interval I let $I_k = I \cap [a_k,b_k]$ and define a function G on the intervals of $[a,b]$ by $G(I) = \sum_k \int_{I_k} f$. If I and J are non-overlapping intervals, then

$$G(I \cup J) = \sum_k \int_{(I\cup J)_k} f = \sum_k \left(\int_{I_k} f + \int_{J_k} f\right)$$

$$= \sum_k \int_{I_k} f + \sum_k \int_{J_k} f$$

$$= G(I) + G(J)$$

since the series are absolutely convergent. Hence, the function G is finitely additive. Note that $G([a,b]) = \sum_k \int_{a_k}^{b_k} f$.

Let $\varepsilon > 0$ and choose N so that $\sum_{k=N+1}^{\infty} \omega(\int_{a_k}^{t} f,[a_k,b_k]) < \varepsilon/4$. For each k choose a positive function δ_k on $[a_k,b_k]$ so that $|g(\mathcal{P}) - G([a_k,b_k])| < \varepsilon\, 2^{-k-2}$ whenever $\mathcal{P}$ is sub δ_k on $[a_k,b_k]$. Define δ on $[a,b]$ as follows:

(i) If $t \in (a_k, b_k)$, then $\delta(t) = \min \{\delta_k(t), t - a_k, b_k - t\}$.

(ii) If $t \in E - \bigcup_k \{a_k, b_k\}$, then choose $\delta(t) > 0$ so that $(t - \delta(t), t + \delta(t)) \cap [a_j, b_j] = \emptyset$ for $1 \leq j \leq N$.

(iii) If $t = a_k$ and if t is not an isolated point of E, then choose $\delta(t) > 0$ so that $\delta(t) \leq \min \{\delta_k(t), b_k - t\}$ and $(t - \delta(t), t) \cap [a_j, b_j] = \emptyset$ for $1 \leq j \leq N$.

(iv) If $t = b_k$ and if t is not an isolated point of E, then choose $\delta(t) > 0$ so that $\delta(t) \leq \min \{\delta_k(t), t - a_k\}$ and $(t, t + \delta(t)) \cap [a_j, b_j] = \emptyset$ for $1 \leq j \leq N$.

(v) If $a_{k_1} = t = b_{k_2}$, then $\delta(t) = \min\{\delta_{k_1}(t), b_{k_1} - t, \delta_{k_2}(t), t - a_{k_2}\}$.

This defines δ on $[a,b]$.

Let $\mathcal{P}$ be sub δ on $[a,b]$ and assume without loss of generality that all of the tags are endpoints. Let $\mathcal{P}_k$ be the subset of $\mathcal{P}$ that has intervals in $[a_k, b_k]$ and note that $\mathcal{P}_k$ is sub δ_k. Let $\mathcal{P}_E = \mathcal{P} - \bigcup_k \mathcal{P}_k$ and note that $I \cap (a_k, b_k) = \emptyset$ for $1 \leq k \leq N$ if I is an interval in $\mathcal{P}_E$. In addition, if $k > N$, then $[a_k, b_k]$ intersects at most two intervals in $\mathcal{P}_E$. Let $\pi = \{k : \mathcal{P}_k \neq \emptyset\}$ and use Henstock's Lemma to compute

$$|g(\mathcal{P}) - G([a,b])| = |g(\mathcal{P}_E) + \sum_\pi g(\mathcal{P}_k) - G(\mathcal{P}_E) - \sum_\pi G(\mathcal{P}_k)|$$

$$\leq \sum_\pi |g(\mathcal{P}_k) - G(\mathcal{P}_k)| + |G(\mathcal{P}_E)|$$

$$\leq \sum_\pi \varepsilon 2^{-k-2} + 2 \sum_{k=N+1}^{\infty} \omega(\int_{a_k}^{t} f, [a_k, b_k]).$$

By the choice of N we find that $|g(\mathcal{P}) - G([a,b])| < \varepsilon$. Hence, the function g is GR integrable on $[a,b]$ and $\int_a^b g = \sum_k \int_{a_k}^{b_k} f$. This completes the proof.

The functions U_δ and V_δ that appear in the next definition correspond to the major and minor functions in Perron integration. However, as we shall see, the full power of major and minor functions is not required.

DEFINITION 13: Let $f : [a,b] \longrightarrow \mathbb{R}$ and let δ be a positive function defined on $[a,b]$. For each $t \in (a,b]$ define

$$U_\delta(t) = \sup \{f(\mathcal{P}) : \mathcal{P} \text{ is sub } \delta \text{ on } [a,t]\}$$

and

$$V_\delta(t) = \inf \{f(\mathcal{P}) : \mathcal{P} \text{ is sub } \delta \text{ on } [a,t]\},$$

the values $+\infty$ and $-\infty$ being allowed. For completeness let $U_\delta(a) = 0 = V_\delta(a)$.

If f is GR integrable on $[a,b]$, then it follows from Definition 6 and Lemma 7 that there exists a positive function δ on $[a,b]$ such that U_δ and V_δ are finite-valued. The next lemma lists some of the properties of U_δ and V_δ. The proof is based upon the definition of the GR integral and is not difficult.

LEMMA 14: Let $f : [a,b] \longrightarrow \mathbb{R}$ be GR integrable on $[a,b]$ and let δ be a positive function defined on $[a,b]$ for which U_δ and V_δ are finite-valued.

(a) If $[c,d] \subset [a,b]$, then

$$U_\delta(d) - U_\delta(c) \geq \sup\{f(\mathcal{P}) : \mathcal{P} \text{ is sub } \delta \text{ on } [c,d]\} \geq \int_c^d f$$

and

$$V_\delta(d) - V_\delta(c) \leq \inf\{f(\mathcal{P}) : \mathcal{P} \text{ is sub } \delta \text{ on } [c,d]\} \leq \int_c^d f.$$

(b) If $F(t) = \int_a^t f$, then the functions $F - U_\delta$ and $V_\delta - F$ are nonincreasing on $[a,b]$.

Let E be a closed subset of $[a,b]$ with bounds c and d and let $f : E \longrightarrow \mathbb{R}$. Let $[c,d] - E = \bigcup_n (c_n, d_n)$ where the intervals $\{(c_n, d_n)\}$ are the intervals contiguous to E. Define $g : [a,b] \longrightarrow \mathbb{R}$ as follows:

(i) If $t \in [a,c]$, then $g(t) = f(c)$.

(ii) If $t \in [d,b]$, then $g(t) = f(d)$.

(iii) If $t \in E$, then $g(t) = f(t)$.

(iv) If $t \in (c_n, d_n)$, then $g(t) = \dfrac{f(d_n)-f(c_n)}{d_n - c_n}(t - c_n) + f(c_n)$.

In this case we say that g is the function that equals f on E and is linear on the intervals contiguous to E. The next three lemmas are concerned with such functions. The proofs, although not difficult, are tedious and will be omitted.

LEMMA 15: Let E be a closed set with bounds a and b and let $f : [a,b] \longrightarrow \mathbb{R}$ be BV_* on E. If $g : [a,b] \longrightarrow \mathbb{R}$ is the function that equals f on E and is linear on the intervals contiguous to E, then g is BV_* on $[a,b]$.

LEMMA 16: Let f and g be functions defined on $[a,b]$ and let E be a closed subset of $[a,b]$. Let f_1 and g_1 be the functions that equal f and g on E, respectively, and are linear on the intervals contiguous to E. If $f - g$ is nonincreasing on E, then $f_1 - g_1$ is nonincreasing on $[a,b]$.

LEMMA 17: Let E be a closed subset of $[a,b]$ and let $f : E \longrightarrow \mathbb{R}$. Let g be the function that equals f on E and is linear on the intervals contiguous to E. If t_0 is both a right-hand and left-hand limit point of E and if f has a derivative with respect to E at t_0, then g is differentiable at t_0 and $g'(t_0) = f'_E(t_0)$, where $f'_E(t_0)$ is the derivative of f with respect to E at t_0.

We are now in a position to prove that the indefinite GR integral is ACG_*.

THEOREM 18: Let $f : [a,b] \longrightarrow \mathbb{R}$ be GR integrable on $[a,b]$. If $F(t) = \int_a^t f$, then F is ACG_* on $[a,b]$.

Proof: We will use Theorem 2 to prove that the continuous function F is ACG_* on $[a,b]$. Let E be a perfect set in $[a,b]$. Since F is BVG_* on $[a,b]$ there exists an interval $[c,d] \subset [a,b]$ such that $c,d \in E$, $E \cap (c,d) \neq \emptyset$, and F is BV_* on $E \cap [c,d]$ (Theorem 2). Let G be the function that equals F on $E \cap [c,d]$ and is linear on the intervals contiguous to $E \cap [c,d]$. By Lemma 15 the function G is BV_* on $[c,d]$. To complete the proof it is sufficient to prove that G is AC on $[c,d]$.

For then F is AC on $E \cap [c,d]$ and by Theorem 3 it follows that F is AC_* on $E \cap [c,d]$.

To this end let $[c,d] - E = \bigcup_n (c_n,d_n)$ and define $g : [c,d] \longrightarrow \mathbb{R}$ by

$$g(t) = \begin{cases} \dfrac{F(d_n)-F(c_n)}{d_n-c_n} & \text{if } t \in (c_n,d_n) \\ \\ f(t) & \text{if } t \in E \cap [c,d] \end{cases}$$

Then by Theorem 10, Lemma 17, and the definition of G we see that $G' = g$ almost everywhere on $[c,d]$. Since G is BV_* (and hence BV) on $[c,d]$ the function g is Lebesgue integrable on $[c,d]$ and it follows that f is Lebesgue integrable on $E \cap [c,d]$. Define $\emptyset : [c,d] \longrightarrow \mathbb{R}$ by $\emptyset(t) = G(t) - G(c) - \int_c^t g$. To prove that G is AC on $[c,d]$ it is sufficient to prove that $\emptyset(t) = 0$ for all t in $[c,d]$.

Let $\varepsilon > 0$ and choose a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - F(b)| < \varepsilon$ whenever $\mathcal{P}$ is sub δ on $[a,b]$ and let $U(t) = U_\delta(t)$ and $V(t) = V_\delta(t)$. Let U_1 and V_1 be the functions defined on $[c,d]$ that equal U and V, respectively, on $E \cap [c,d]$ and are linear on the intervals contiguous to $E \cap [c,d]$. By Lemma 14(b) the functions $F - U$ and $V - F$ are nonincreasing on $[a,b]$ and by Lemma 16 the functions $G - U_1$ and $V_1 - G$ are nonincreasing on $[c,d]$.

The next step is to prove that

$$V_1(t) - V_1(c) \le (L)\int_c^t g \le U_1(t) - U_1(c) \tag{4}$$

for each t in $[c,d]$. Now $\sum_n \omega(\int_{c_n}^t f, [c_n,d_n]) < \infty$ since F is BV_* on $E \cap [c,d]$. Furthermore, the function f is GR integrable on $E \cap [c,d]$ since it is Lebesgue integrable on $E \cap [c,d]$. Hence, the hypotheses of Theorem 12 are satisfied. There are two cases to consider.

(i) If $t \in E \cap [c,d]$, then

$$(GR)\int_c^t f = (L)\int_c^t f\chi_E + \sum_{d_n \leq t} (GR)\int_{c_n}^{d_n} f = (L)\int_c^t g\chi_E + \sum_{d_n \leq t} (L)\int_{c_n}^{d_n} g = (L)\int_c^t g.$$

By Lemma 14(a) we obtain

$$V_1(t) - V_1(c) = V(t) - V(c) \leq (GR)\int_c^t f = (L)\int_c^t g \leq U(t) - U(c) = U_1(t) - U_1(c).$$

(ii) If $t \in (c_k,d_k)$, then using (i) and Lemma 14(a) we obtain

$$(L)\int_c^t g = (GR)\int_c^{c_k} f + \frac{F(d_k)-F(c_k)}{d_k-c_k}(t - c_k)$$

$$\leq U(c_k) - U(c) + \frac{U(d_k)-U(c_k)}{d_k-c_k}(t - c_k)$$

$$= U_1(t) - U_1(c)$$

and

$$(L)\int_c^t g = (GR)\int_c^{c_k} f + \frac{F(d_k)-F(c_k)}{d_k-c_k}(t - c_k)$$

$$\geq V(c_k) - V(c) + \frac{V(d_k)-V(c_k)}{d_k-c_k}(t - c_k)$$

$$= V_1(t) - V_1(c).$$

Thus, equation (4) is established.

Since $V(c) \leq F(c) \leq U(c)$ by Lemma 14(a), since $-\varepsilon \leq F(d) - U(d) \leq F(d) - V(d) \leq \varepsilon$ by Henstock's Lemma, and since the functions $G - U_1$ and $V_1 - G$ are nonincreasing on $[c,d]$ we obtain

$$\begin{aligned} \emptyset(t) &\geq \emptyset(t) - [U_1(t) - U_1(c) - (L)\int_c^t g] \\ &= G(t) - U_1(t) + U_1(c) - G(c) \\ &\geq G(d) - U_1(d) + U_1(c) - G(c) \\ &= F(d) - U(d) + U(c) - F(c) \\ &\geq -\varepsilon \end{aligned}$$

and

$$\begin{aligned} \emptyset(t) &\leq \emptyset(t) + [(L)\int_c^t g - (V_1(t) - V_1(c))] \\ &= G(t) - V_1(t) + V_1(c) - G(c) \\ &\leq G(d) - V_1(d) + V_1(c) - G(c) \\ &= F(d) - V(d) + V(c) - F(c) \\ &\leq \varepsilon. \end{aligned}$$

These inequalities are valid for each t in $[c,d]$. Hence, we see that $|\emptyset(t)| \leq \varepsilon$ for all t in $[c,d]$. Since $\varepsilon > 0$ was arbitrary we conclude that $\emptyset(t) = 0$ for all t in $[c,d]$. This completes the proof.

Theorems 10 and 18 prove that every GR integrable function is D_* integrable and that the integrals are equal. Thus, the converse of Theorem 8 has been established.

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