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AN L¹ VERSION OF THE GAUSS INTEGRAL THEOREM

1.0 Introduction

This paper presents a generalized version of Gauss' integral theorem in the form:

$$\int_{\partial(x+R)} u(s)\xi_k(s)ds = \int_{x+R} \frac{\partial u}{\partial y_k}(y)dy \qquad (1)$$

for all $x \in \mathbb{R}^n$.

Here ξ_k is the k^{th} component of the outer unit normal to the region R translated by a point $x \in \mathbb{R}^n$. The usual version of this theorem requires u to be C^1 on the closure of some fixed $x + R$ and that the boundary of $x + R$ be composed of finitely many C^1 simple hypersurfaces. See Kellogg [6] for details. In this case ds is the usual element of surface area defined in advanced calculus. (see (3) below).

More general versions of this theorem have been found by numerous authors. For example for $n = 2$, the papers of Verblunsky [11], Potts [8], and Menger [7] give extensions of Greens theorem in the plane, to which (1) is equivalent. The article by Gray and Morris [4] gives a good bibliography of work along these lines and surveys some of the results. As an example, let C be a simple closed contour in the plane with K the closure of the bounded region determined by C. Let P, Q be real valued functions on $\mathbb{R}^2$ satisfying:

1. $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ exist and are continuous on K
2. P, Q are continuous on K.
3. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \in L^1(K)$.

Then

$$\int_C P\,dx + Q\,dy = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy .$$

Another sort of generalization of Gauss' theorem aims at reducing the hypotheses on the integrands in question with price of stronger hypotheses on R. For example, again in $\mathbb{R}^2$, a theorem of P.J. Cohen [1] states that hypotheses (1) (2) (3) above may be replaced by:

(1') $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ exist on K

(2) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \in L'(K)$

(3) P, Q continuous on K

when K is a rectangle.

The purpose of this paper is to present a result in $\mathbb{R}^n$ which reduces the hypotheses on both u and R in (1). A function $u \in L^1(\mathbb{R}^n)$ is said to have an L^1 first partial derivative with respect to x_k if there is a function $u_{x_k} \in L^1(\mathbb{R}^n)$ satisfying for $\bar{h}_k = (0,\dots,0,h_k,0,\dots,0)$:

$$\lim_{h_k \to 0} \int_{\mathbb{R}^n} \left| \frac{u(x + \bar{h}_k) - u(x)}{h_k} - u_{x_k} \right| dx = 0$$

Our version of (1) will require u to have L^1 first partial derivatives.

2.0 Gauss' theorem as a convolution equation

When f is an L^1 function and μ is a finite Borel measure on $\mathbb{R}^n$, the convolution $f * \mu$ is defined for almost all x by:

$$(f * \mu)(x) = \int_{\mathbb{R}^n} f(x - t)\, d\mu(t)$$

The integrals in (1) can be recognized as convolutions with appropriate measures. This idea is behind much recent work on reconstructing functions from knowledge of their averages over hyperplanes and spheres. See the survey of Zalcman [12] for a discussion of some of these interesting results.

For the present, the region R will be assumed to satisfy the following hypotheses

(i) ∂R is a simple closed C^1 curve

(ii) $0 \in \text{int}(R)$ (interior of R) (2)

(iii) the surface measure ds on ∂R is given by:

$$ds = |\xi(u,v)|du\,dv, \quad \text{where} \quad \xi(u,v)$$

is the outer unit normal to the point on ∂R specified by parameters u,v in a C^1 parametrization. These hypotheses will be substantially reduced below.

Before the first proof of (1) is given, a simple lemma (See Stein and Weiss [10], p. 4) will be provided for completeness. We define the Fourier transform of an L^1 function g by:

$$\hat{g}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot t}\, g(t)dt \ .$$

__Lemma 1__. If $f \in L^1(\mathbb{R}^n)$ and f_{x_k} is its L^1 partial derivative, then

$$\hat{f}_{x_k} = 2\pi i x_k \hat{f}(x) \ .$$

__Proof__:

$$\left|\int e^{-2\pi i x\cdot t}\, f_{x_k}(t)dt - \int e^{-2\pi i x\cdot t}\, \frac{[f(t-\bar{h}_k)-f(t)]}{h_k}\, dt\right|$$

$$\leq \int \left|f_{x_k}(t) - \frac{[f(t-\bar{h}_k)-f(t)]}{h_k}\right| dt \to 0$$

$$\text{as} \quad h_k \to 0 \ .$$

Hence by the translation properties of the Fourier transform:

$$\left|\hat{f}_{x_k}(x) - \hat{f}(x)\ [\frac{e^{2\pi i h\cdot x}\ 1]}{h_k}\right| \to 0$$

i.e.

$$\hat{f}_{x_k}(x) = 2\pi i x_k \hat{f}(x) \ .$$

So as to be able to consider the normal component $\xi_k(s)$ as a signed density function, we define:

$$\xi_k(x + R,\ x + s) = \begin{cases} k^{th} \text{ component of unit outer normal to } x + R \text{ at } x + s & \text{if } s \in \partial R \\ 0\ , & \text{if } s \notin \partial R \end{cases}$$

Then $\xi_k(x + R,\ x + s)$ is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ which by (2) is continuous on $\mathbb{R}^n \times \partial R$. Also note that, from geometry

$$\xi_k(x + R,\ x + s) = \xi_k(R,s)$$
$$\xi_k(R,-s) = -\xi_k(-R,s) \tag{3}$$

To write (1) as a convolution equation, let μ be Lebesgue measure on $-R = \{-x\colon x \in R\}$, and let μ_k be the signed measure:

$$d\mu_k = -\xi_k(-R,t)\chi_{\partial(-R)}(t)dt$$

with dt denoting Lebesgue measure on $\mathbb{R}^n$. There follows:

$$\begin{aligned}\int_{x+R} u_{y_k}(t)dt &= \int u_{y_k}(t)\chi_{x+R}(t)dt\\ &= \int u_{y_k}(t)\chi_{-R}(x - t)dt\\ &= \int u_{y_k}(x - t)\chi_{-R}(t)dt\\ &= (u_{y_k}*\mu)(x)\ .\end{aligned}$$

The surface integral in (1) becomes:

$$\begin{aligned}\int_{\partial(x+R)} u(s)\xi_k(s)ds &= \int u(t)\xi_k(x + R,t)\chi_{\partial(x+R)}(t)dt\\ &= \int u(t)\xi_k(x + R,t)\chi_{\partial(-R)}(x-t)dt\\ &= \int u(x - t)\xi_k(x + R,x-t)\chi_{\partial(-R)}(t)dt\\ &= \int u(x - t)\xi_k(R,-t)\chi_{\partial(-R)}(t)dt\\ &= -\int u(x - t)\xi_k(-R,t)\chi_{\partial(-R)}(t)dt\\ &= (u*\mu_k)(x)\ .\end{aligned}$$

Now an L^1 version of Gauss' theorem can be proved.

__Theorem 1__. Let the region R satisfy the conditions (2). If $u \in L^1(\mathbb{R}^n)$ has L^1 first partial derivatives u_{y_k} then for almost all $x \in \mathbb{R}^n$;

$$\int_{x+R} u_{y_k}(t)dt = \int_{\partial(x+R)} u(s)\xi_k(s)ds$$

Proof:

$$\int_{x+R} u_{y_k}(t)dt = (u_{y_k}*\mu)(x) \quad \text{and}$$

$$(u_{y_k}*\mu)\hat{}(w) = \hat{u}_{y_k}(w)\hat{\mu}(w) = 2\pi i w_k \hat{u}(w) \int e^{-2\pi i w\cdot t} d\mu(t)$$

$$= 2\pi i w_k \hat{u}(w) \int_{-R} e^{-2\pi i w\cdot t} dt$$

$$= -\hat{u}(w) \int_{-R} \frac{\partial}{\partial t_k} e^{-2\pi i w\cdot t} dt$$

$$\overset{(*)}{=} -\hat{u}(w) \int e^{-2\pi i w\cdot t} \xi_k(-R,t)\chi_{\partial(-R)}(t)dt .$$

The last equality follows from Gauss' theorem for C^1 functions on $\bar{R}$. To complete the proof, notice that the Fourier transform of the surface integral is:

$$(\int_{\partial(x+R)} u(s)\xi_k(s)ds)\hat{}(w) = (u*\mu_k)\hat{}(w)$$

$$= -\hat{u}(w) \int e^{-2\pi i w\cdot t} \xi_k(-R,t)\chi_{\partial(-R)}(t)dt .$$

Hence $(u_{y_k}*\mu)\hat{}(w) = (u*\mu_k)\hat{}(w)$ for all w, and consequently $(u_{y_k}*\mu)(x) = (u*\mu_k)(x)$ for almost all x.

3.0 Generalizations

Our first goal is to relax the conditions on the region R in (1). An examination of the above proof shows that if a Gauss theorem for the function $t \mapsto \exp(-2\pi i w\cdot t)$ is available for a more general region R, then it can be inserted at the equality marked with the asterisk (*) and (1) will hold for such regions.

The elements to be generalized are the notions of surface measure and unit outer normal. Quite general regions can be admitted in Gauss' theorem provided the vector field in question is Lipschitzian and $n-1$ dimensional Hausdorff measure is considered the surface measure. The most general theorem of this kind known to the writer is due to DeGiorgi and Federer. (See Federer [3].) To state this theorem, a few preliminaries are needed. We follow the treatment in Federer [3] in these matters. For any $S \subset \mathbb{R}^n$ and non-negative real number m, let

$$H^{m,\delta}(S) = \inf \sum_{T \in G} \alpha(m) \left[\frac{\mathrm{diam}(T)}{2}\right]^m$$

where the infimum is taken over all countable coverings G of S with $\mathrm{diam}(T) \leq \delta$ for $T \in G$. The normalization constant $\alpha(m)$ is given by:

$$\alpha(m) = \Gamma(\tfrac{1}{2})^m \,/\, \Gamma(\tfrac{m}{2}+1)$$

The m dimensional Hausdorff (outer) measure of S is defined as:

$$H^m(S) = \lim_{\delta \to 0} H^{m,\delta}(S) \ .$$

From these definitions it is seen that the special cases

$$H^0(S) = \text{counting measure of } S$$

$$H^n(S) = \text{Lebesgue measure of } S$$

obtain. Henceforth we denote Lebesgue measure on $\mathbb{R}^n$ by $\mathcal{L}_n$ to avoid confusion with other measures.

Now let R be any $\mathcal{L}^n$ measurable subset of $\mathbb{R}^n$. An entirely measure theoretic definition can be given for the unit outer normal $\xi(R,b)$ at $b \in \partial R$ in terms of the metric density of restrictions of $\mathcal{L}^n$ (denoted $\mathcal{L}^n/S$).

For any $\mathcal{L}^n$ measurable set S let

$$\theta^{*n}(\mathcal{L}^n/S,b) = \overline{\lim_{r \to 0}} \ \frac{\mathcal{L}^n(B(b,r) \cap S)}{\alpha(n) r^n}$$

where $B(b,r)$ is the ball of radius r centered at b. The density Θ_*^n is defined by replacing $\overline{\lim}$ by $\underline{\lim}$. When $\theta_*^n = \theta^{*n}$ the common value is denoted by θ^n. The exterior unit normal n of R at $b \in \partial R$ is defined to be a vector of unit Euclidean length which satisfies:

$$\theta^n(\mathcal{L}^n/\{x:(x-b)\cdot u > 0\} \cap R, b) = 0$$

$$\theta^n(\mathcal{L}^n/\{x:(x-b)\cdot u < 0\} \cap R^c, b) = 0$$

It can be shown there is at most one such u. Finally, let

$$\begin{aligned} \xi(R,b) &= u \quad \text{if } R \text{ has an exterior unit normal at } b \\ &= 0 \quad \text{otherwise .} \end{aligned}$$

The following extension of Gauss theorem can now be stated. See Federer [3] p. 478 for a proof.

Theorem 2. (DeGiorgi and Federer).

Let R be $\mathcal{L}^n$ measurable and suppose $H^{n-1}(K \cap \partial R) < \infty$ for every compact $K \subset \mathbb{R}^n$. If ξ is a Lipschitzian vector field on $\mathbb{R}^n$ with compact support then

$$\int \frac{\partial}{\partial x_k} \xi(x) d\mathcal{L}^n(x) = \int \xi(x)\xi_k(R,x) dH^{n-1}(x) \ .$$

With these tools Theorem 1 can be improved to the following result.

Theorem 3.

Let $u \in L^1(\mathbb{R}^n)$ have L^1 first partial derivatives u_{y_k}. If R satisfies the hypotheses for Theorem 2 then:

$$\int_{x+R} u_{y_k}(t) d\mathcal{L}^n(t) = \int u(t)\xi_k(x+R,t) dH^{n-1}(t)$$

for $\mathcal{L}^n$ almost all $x \in \mathbb{R}^n$.

Proof.

The measure theoretic normal $\xi_k(R,t)$ can be used to define a function $\xi_n(x+R, x+s)$ on $\mathbb{R}^n \times \mathbb{R}^n$ as before. The translation and notation invariance of $\mathcal{L}^n$ implies that the symmetry conditions (3) are satisfied by ξ_k. This time we define the Borel measure:

$$d\mu_k(t) = -\xi_k(-R,t)\chi_{\partial(-R)}(t) dH^{n-1}(t)$$

and we let the surface integral in (1) be written as

$$\int u(t)\xi_R(R + x,\ x + s)dH^{n-1}(s)$$

$$= \int u(t)\xi_k(x + R, t)\chi_{\partial(x+R)}(t)dH^{n-1}(t)$$

The proof now proceeds exactly as in Theorem 1 with Theorem 2 invoked at the equality marked with an asterisk (*).

Next, a proof of a special case of Theorem 3 will be given which offers different possibilities for generalization. Furthermore this proof doesn't rest on any previous versions of Gauss' theorem. Let $S(x,\alpha)$ be the sphere $\{y:|y - x| = \alpha\}$. If $u \in L^1(\mathbb{R}^n)$ has L^1 first partial derivatives then for fixed $\alpha > 0$,

$$\int_{B(x,\alpha)} u_{y_k}(t)dt = \int_{S(x,\alpha)} u(s)\xi_k(s)ds \qquad (4)$$

for $\mathcal{L}^n$ almost all x.

<u>(Alternative) Proof of (4)</u>:

Define μ and μ_k as before with $R = B(x,\alpha)$. A direct calculation of $\hat{\mu}$ and $\hat{\mu}_k$ (for which see Stein and Weiss [10] p. 153 and Ch. 10 in Erdelyi et al [2] for some necessary Bessel function identities) results in:

$$2\pi i x_k\hat{\mu}(x) = \hat{\mu}_k(x) \qquad (5)$$

using this fact we obtain

$$(u * \mu_k)\hat{} = \hat{u}\hat{\mu}_k = 2\pi i x_k\hat{u}\hat{\mu}$$
$$= \hat{u}_{x_k}\hat{\mu} = (u_{x_k} * \mu)\hat{}\ ,$$

so $(u * \mu_k)(x) = (u_{x_k} * \mu)(x)$ for $\mathcal{L}^n$ almost all x, and (4) is proved.

<u>Remark</u>: The relation (5) expresses the fact that μ_k is the distributional derivative of μ, and can be established for regions with simple bounding hypersurfaces. (See Hörmander [5], p. 60). The ball $B(x,\alpha)$ was chosen as the domain of integration in (4) for the purpose of stating the following problem: Is the Gauss formula (4) true for every $\alpha > 0$ for $\mathcal{L}^n$ almost all x? This is perhaps the simplest case of a possible generalization of Theorem 1 which seeks to allow the integration domains $x + R$ to vary for each x. There are some measure theoretic subtleties to be faced to even make sense of the surface integral in (4) in this simple case.

Since μ_k is a finite Borel measure, the convolution $u * \mu_k$ is defined and can be represented by the surface integral in (4) for $\mathcal{L}^n$ almost all x. However the measure μ_k depends on α, so we can't conclude that the convolution is the surface integral for every $\alpha > 0$, for $\mathcal{L}^n$ almost all x. The following lemma, whose proof can be found in Stein and Wainger [9], gives meaning to the surface integral for every $\alpha > 0$, for $\mathcal{L}^n$ almost all x.

Lemma 2. If $n \geq 3$ and f is $\mathcal{L}^n$ measurable on $\mathbb{R}^n$, then for $\mathcal{L}^n$ almost all x, f restricted to $S(x,\alpha)$ is Lebesgue measurable on $S(x,\alpha)$ for every $\alpha > 0$.

Remarks:

1. Lebesgue measure on $S(x,\alpha)$ is the rotation invariant measure of unit mass on $S(x,\alpha)$.

2. It is not known to the writer whether $n \geq 3$ can be changed to $n \geq 2$ in the above lemmas.

Finally, a suggestion for improving Theorem 1 from an anonymous referee is gratefully acknowledged.

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