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# SOME ANALYSIS WITHOUT COVERING THEOREMS

Let $f$ be a function of bounded variation on the interval $[a,b]$ and for $a \leq x \leq b$ let $t(x)$ be the total variation of $f$ on $[a,x]$. Let $E$ denote the set of all points $x \in (a,b)$ such that either $f$ or $t$ has no finite or infinite derivative, or $f$ and $t$ have derivatives at $x$ and $t'(x) > |f'(x)|$. Let $m$ denote the Lebesgue measure and $m_e$ denote the Lebesgue outer measure. A theorem attributed to de la Vallée Poussin [3, (9.2), (9.6) (ii) pp. 125, 127] states:

**Theorem 1** (de la Vallée Poussin). Let $f$, $t$ and $E$ be as above. Then $mt(E) = m(E) = 0$.

The equation $mt(E) = 0$ is harder to prove than $m(E) = 0$, in part because $t$ might map sets of measure $0$ to sets of positive measure. Nevertheless, the former equation is important in real function theory. Witness chapters VII and IX of [3].

Now let $E$ be any subset of $(a,b)$. We say that $x$ is a right (left) point of density of $E$ if $\lim_{h \downarrow 0} m_e((x,x+h) \cap E) = 1$ ($\lim_{h \downarrow 0} m_e((x-h,x) \cap E) = 1$). We say that $x$ is a point of density of $E$ if it is both a left and right point of density of $E$. A well-known "density" theorem states:

**Theorem 2.** Almost all points of any set $E$ are points of density of $E$.

It is known that if $f$ is an arbitrary real valued function on $[a,b]$, the set of points at which $f$ has an infinite derivative is a set of measure $0$. In [3, (4.4), p. 270] the contingent of plane sets is used to generalize this statement. We generalize again to obtain:

**Theorem 3.** Let $E \subset (a,b)$ be a nonvoid set and let $f$ be a real valued function defined on $E$ such that for each $x \in E$ which is a right accumulation point of $E$, we have

$$\lim_{u \in E,\ u \downarrow x} |f(u) - f(x)|(u - x)^{-1} = \infty .$$

Then $m(E) = 0$.

Theorems 1, 2 and 3 can be proved by means of results that depend on the Vitali covering theorem. Sometimes Sierpinski's covering theorem is applied (consult [4, (11.41)] and what follows). The proof of Theorem 1 in [3] uses, moreover, integration of derivatives and other results that take time to develop. Theorem 2 can also be proved by arguments that are much like Banach's proof of the Vitali covering theorem [2, Theorem 3.20]; in [5], for a measurable $E$, a proof is given that does not require covering theorems. In this note we give relatively simple proofs of Theorems 1, 2 and 3 where no covering theorems, integration or contingents of plane sets are used. Our proof of Theorem 2 is different from the proof in [5].

Here is one principle we will use repeatedly explicitly and implicitly. If $S_1 \subset S_2 \subset S_3 \subset \cdots$ is an expanding sequence of sets, then $\lim m_e(S_n) = m_e(\cup_n S_n)$.

We begin with a key lemma whose order of difficulty is roughly the same as the Vitali covering theorem.

**Lemma 1.** Let $F$ and $f$ be nondecreasing functions on $[a,b]$ and let $E \subset [a,b)$. Suppose that for each $x \in E$ there exists a sequence of positive numbers $(h_n)$ converging to $0$ such that $F(x + h_n) - F(x) \geq f(x + h_n) - f(x)$ for each $n$. Then $m_eF(E) \geq m_ef(E)$.

**Proof.** Let $c$ and $d$ be numbers with $0 < c < d < (b-a)/2$. Let

$$E_c = \{x \in E\colon \text{there is an } h \in [c,d] \text{ such that } F(x + h) - F(x) \geq f(x + h) - f(x)\} ,$$

$$X = \{x \in [a,b)\colon m_ef((a,x) \cap E_c) \leq F(x+) - F(a+)\}.$$

Then $a \in X$. Let $v = \sup X$.

We claim that $v \geq b - d$. To prove it, assume that $v < b - d$. It follows routinely that $v \in X$ and that $v$ is a right accumulation point of $E_c$. Let $(w_n)$ be a decreasing sequence of points in $E_c$ converging to $v$. There are numbers $h_n \in [c,d]$ such that $F(w_n + h_n) - F(w_n) \geq f(w_n + h_n) - f(w_n)$ for each $n$. Without loss of generality, we may assume that the sequence $(h_n)$ converges; let $H = \lim h_n$. Then $c \leq H \leq d$ so that $v < v + H < b$. There is a nonincreasing sequence $(k_n)$ and a nondecreasing sequence $(k_n')$ such that $\lim k_n = \lim k_n' = v + H$ and that $k_n' \leq w_n + h_n \leq k_n$ for each $n$. Then for each sufficiently great $n$, $F(k_n+) - F(v+) \geq F(w_n + h_n) - F(w_n) \geq f(w_n + h_n) - f(w_n) \geq m_ef((v,v + H) \cap E_c)$. Hence $F((v + H)+) - F(v+) \geq m_ef((v,v+H) \cap E_c)$. This, together with $F(v+) - F(a+) \geq m_e((a,v) \cap E_c)$, implies that

$$F((v + H)+) - F(a+) \geq m_ef((a,v+H) \cap E_c) .$$

So $v + H \in X$. This contradiction proves that $v \geq b - d$.

Now we choose an $x \in X$ with $x > b - 2d$. Then $F(b-) - F(a+) \geq F(x+) - F(a+) \geq m_ef((a,x) \cap E_c) \geq m_ef((a,b - 2d) \cap E_c)$. Letting first $c$ and then $d$ approach 0, we obtain $F(b-) - F(a+) \geq m_ef(E)$.

Let $U$ be any open set containing $F(E)$ and let $I_1, I_2, I_3, \ldots$ be the components of $U$. Each $F^{-1}(I_n)$ is a subinterval of $[a,b]$, and we obtain from the preceding paragraph $m_ef(E \cap F^{-1}(I_n)) \leq m(I_n)$ for each $n$. Therefore

$$m_ef(E) \leq \sum_n m_ef(E \cap F^{-1}(I_n)) \leq \sum_n m(I_n) = m(U) .$$

Finally, we approximate $m_eF(E)$ with $m(U)$ and obtain $m_ef(E) \leq m_eF(E)$. □

Lemma 1 is essentially all we need to prove Theorem 1.

**Proof of Theorem 1.** Our strategy is to prove it first for right derivatives. The result for left derivatives is found by substituting $-t(a + b - x)$ for $t(x)$ and $-f(a + b - x)$ for $f(x)$. Finally, there are at most countably many points where a function has left and right derivatives but has no two-sided derivative [1, (17.9)]. We proceed in three steps.

1. Let $S$ be the set of all points $x$ such that $t$ has no right derivative, finite or infinite, at $x$. We claim that $m_et(S) = 0$. To prove this, choose rational numbers $c$ and $d$ with $c > d$ and define

$$S_0 = \{x \in S: D^+t(x) > c,\ d > D_+t(x)\}\ .$$

Applying Lemma 1 to $F = t$ and $f(x) = cx$, we obtain $m_e t(S_o) \geq cm_e(S_o)$; with $F(x) = dx$ and $f = t$ we get similarly $m_e t(S_o) \leq dm_e(S_o)$. Hence $m_e(S_o) = m_e t(S_o) = 0$. It follows that $m(S) = mt(S) = 0$.

2. We prove that $m_e t(E) = 0$. Assume to the contrary that $m_e t(E) > 0$. By part 1 we have also $m_e t(E\backslash S) > 0$. Each point $x \in E\backslash S$ satisfies the strict inequalities $t'_+(x) > D_+f(x)$ and $t'_+(x) > -D^+f(x)$. It follows that there is a $c \in (0,1)$ such that $m_e t(E_o) > 0$ where

$$E_0 = \{x \in E:\ ct'_+(x) > D_+f(x) \text{ and } ct'_+(x) > -D^+f(x)\}\ .$$

Let $a = u_0 < u_1 < \cdots < u_n = b$ be a partition of $[a,b]$ such that

$$(1)\qquad t(b) - t(a) - \Sigma_{j=1}^{n} |f(u_j) - f(u_{j-1})| < (1 - c)m_e t(E_0)\ .$$

Let $\alpha_j = -1$, if $f(u_{j-1}) < f(u_j)$, and $\alpha_j = 1$ otherwise $(j = 1,\ldots,n)$. Let $g$ be a function on $[a,b]$ such that

$$g(x) = t(x) - \Sigma_{i=1}^{j-1} |f(u_j) - f(u_{j-1})| + \alpha_j(f(x) - f(u_{j-1}))$$

for $x \in [u_{j-1},u_j]$ $(j = 1,\ldots,n)$. It is easy to see that $g$ is a nondecreasing function on $[a,b]$ and that $g(b) - g(a)$ equals the left-hand side of (1). Hence $g(b) - g(a) < (1 - c)m_e t(E_o)$.

It follows from the definition of $E_0$ that for each $x \in E_0$ there exist two sequences of positive numbers, $(h_n)$ and $(k_n)$, each converging to 0, such that

$$ct(x + h_n) - ct(x) > f(x + h_n) - f(x)$$

and

$$ct(x + k_n) - ct(x) > -f(x + k_n) + f(x)$$

and hence either

$$g(x + h_n) - g(x) > (1 - c)(t(x + h_n) - t(x))$$

or

$$g(x + k_n) - g(x) > (1 - c)(t(x + k_n) - t(x))$$

for all $n$ for which $x$, $x + h_n$ and $x + k_n$ lie in the same subinterval $[u_{j-1}, u_j]$. By Lemma 1, $m_e g(E_0) \geq (1 - c) m_e t(E_0)$. But

$$m_e g(E_0) \leq g(b) - g(a) < (1 - c) m_e t(E_0) .$$

This contradiction proves that $m_e t(E) = 0$.

3. We prove that $m_e(E) = 0$. Now $E = \bigcup_{c>0} E_c$ where $E_c = \{x \in E: D^+ t(x) > c\}$. It suffices to prove that $m_e t(E_c) = 0$ for any $c > 0$. Applying Lemma 1 to $F = t$ and $f(x) = cx$ we obtain $m_e t(E_c) \geq c m_e(E_c)$. But $m_e t(E_c) = 0$ by part 2, so $m_e(E_c) = 0$. This completes the proof. □

In Theorem 1 we see that if $f$ is a function of bounded variation on $[a,b]$ then $f'$ exists almost everywhere on $[a,b]$. In fact $f'$ is finite almost everywhere on $[a,b]$ [1, (17.17)], and this can be deduced from Lemma 1 as follows. Let $c$ be a positive number and let $S = \{x: t'_+(x) = \infty\}$. Apply Lemma 1 to $F = t$ and $f(x) = cx$ to obtain $m_e t(S) \geq c m_e(S)$. But $c$ is arbitrary, so $m_e(S) = 0$.

We also have $mf(E) = 0$ in Theorem 1. To prove this, let $V$ be an open set containing $t(E)$ and let $I_1, I_2, \ldots$ be the components of $V$. Then $t^{-1}(I_n)$ is a subinterval of $[a,b]$ and

$$\sup f(t^{-1}(I_n)) - \inf f(t^{-1}(I_n)) \leq m(I_n) .$$

Hence $m_e f(t^{-1}(I_n)) \leq m(I_n)$. Finally,

$$m_e f(E) \leq \sum_n m_e(E \cap t^{-1}(I_n)) \leq \sum_n m(I_n) = m(V) .$$

Since $mt(E) = 0$ we have also $mf(E) = 0$.

It is worth noting that Lemma 1 is an easy consequence of the Vitali covering theorem. To see this, let $U$ be an open set containing $F(E)$. Without loss of generality, we delete from $E$ any point where $F$ or $f$ is not continuous. Use the Vitali covering theorem to cover almost all of $f(E)$ with mutually disjoint intervals of the form $[f(x), f(x + h)]$ where $[F(x), F(x + h)] \subset U$ and $F(x + h) - F(x) \geq f(x + h) - f(x)$. Then

$$m_e f(E) \leq \sum (f(x + h) - f(x)) \leq \sum (F(x + h) - F(x)) \leq m(U) ,$$

so $m_e f(E) \leq m_e F(E)$.

Thus a relatively short development of Theorem 1 from the Vitali covering theorem goes as follows. Prove Lemma 1 as in the preceding paragraph, then prove Theorem 1 as we did here.

**Proof of Theorem 2.** We will prove it only for right density. The proof for left density, then, is obtained by considering the set $\{-x: x \in E\}$.

Define the function $f(x) = m_e((a,x) \cap E)$ for $a \leq x \leq b$. It suffices to prove that $f'_+ = 1$ almost everywhere on $E$. But $D^+f(x) \leq 1$ for any $x$, so it suffices to prove that $m_e(S_c) = 0$ where $S_c = \{x \in E: D_+f(x) < 1 - c\}$, for any positive number $c$.

Put $g(x) = x - f(x)$ for $a \leq x \leq b$. Then $g$ is nondecreasing on $[a,b]$ and $D^+g(x) > c$ on $S_c$. By Lemma 1, $m_eg(S_c) \geq cm_e(S_c)$. But $g(b) - g(a) = b - a - m_e(E)$, so $cm_e(S_c) \leq b - a - m_e(E)$.

Now let $U$ be any open set containing $E$ and let $I_1, I_2, \ldots$ be the components of $U$. By the preceding paragraph,

$$cm_e(I_n \cap S_c) \leq m(I_n) - m_e(I_n \cap E)$$

for all $n$, and

$$cm_e(S_c) = \Sigma_n cm_e(I_n \cap S_c) \leq \Sigma_n m(I_n) - \Sigma_n m_e(I_n \cap E) = m(U) - m_e(E) .$$

We approximate $m_e(E)$ with $m(U)$ and obtain $m_e(S_c) = 0$. □

We include a lemma that is so elementary that it requires neither the Vitali covering theorem nor our Lemma 1. We state it for the sake of completeness.

**Lemma 2.** Let $E \subset [a,b]$, let $c$ be a positive number, and let $f$ be a function defined on $E$ such that each $x \in E$ which is a right accumulation point of $E$ satisfies $\limsup_{u \downarrow x,\, u \in E} |f(u) - f(x)|(u - x)^{-1} < c$. Then $m_ef(E) \leq cm_e(E)$.

**Proof.** Fix a number $d > 0$. Let

$$E_d = \{x \in E: |f(u) - f(x)| < c(u - x) \text{ for any } u \in E \cap (x,x + d)\} .$$

Let $U$ be any open set containing $E_d$ and let $I_1, I_2, \ldots$ be the components of $U$. Partition $[a,b]$ into finitely many disjoint intervals $J_1$, $J_2, \ldots,$ each of length $< d$. Now if $u, v \in E_d \cap I_i \cap J_j$, then $|u - v| < d$

and hence $|f(u) - f(v)| < c|u - v|$. Thus it follows that

$$\sup f(E_d \cap I_i \cap J_j) - \inf f(E_d \cap I_i \cap J_j) \leq cm(I_i \cap J_j) ,$$

and hence $m_e f(E_d \cap I_i \cap I_j) \leq cm(I_i \cap J_j)$.

It follows that

$$m_e f(E_d) \leq \Sigma_{ij} m_e f(E_d \cap I_i \cap J_j) \leq \Sigma_{ij} cm(I_i \cap J_j) = cm(U) .$$

We approximate $m_e(E)$ with $m(U)$ and obtain $m_e f(E_d) \leq cm_e(E)$. Letting $d$ tend to $0$ we obtain $m_e f(E) \leq cm_e(E)$. □

Thus if $f$ is defined on $[a,b]$ and $f$ is differentiable at each point of $E$ and $|f'_+(x)| < c$ for $x \in E$, then $m_e f(E) \leq cm_e(E)$. See also [1, (17.27)]. Lemma 2 is what we need to prove Theorem 3.

**Proof of Theorem 3.** It suffices to let $f$ be bounded on $E$ because $E = \cup_{n=1}^{\infty} f^{-1}(-n,n)$. For each integer $N > 0$, let

$$E_N = \{x \in E: |f(u) - f(x)| > u - x \text{ for all } u \in E \cap (x, x + N^{-1})\} .$$

It suffices to prove $m_e(E_N) = 0$, because $E = \cup_{N=1}^{\infty} E_N$. We may suppose that $b - a < N^{-1}$ because the interval $[a,b]$ is the union of finitely many intervals of length $< N^{-1}$.

So we may assume, without loss of generality, that $|f(u) - f(v)| > |u - v|$ for $u, v \in E_N$. Let $g$ be the inverse function $f^{-1}$ of $f$ from $f(E_N)$ to $E_N$. For positive numbers $c$ and $d$, put

$$S_{cd} = \{x \in E_N: |f(u) - f(x)| > c(u - x) \quad \text{for} \quad u \in E_N \cap (x, x + d)\} .$$

If $r, s \in f(S_{cd})$, then $|g(r) - g(s)| < |r - s|$; if furthermore $|g(r) - g(x)| < d$, then $c^{-1}|r - s| > |g(r) - g(s)|$. By Lemma 2,

$$c^{-1} m_e f(E) \geq c^{-1} m_e f(S_{cd}) \geq m_e g(f(S_{cd})) = m_e(S_{cd}).$$

Letting $d$ tend to $0$ we obtain $c^{-1} m_e f(E) \geq m_e(E_N)$. But $f(E)$ is bounded. So let $c$ tend to $\infty$ and obtain $m_e(E_N) = 0$. □

In particular, if $f$ is defined on $[a,b]$ and if $f'(x) = \pm\infty$ for each $x \in E$, then $m(E) = 0$. See also [3, (4.4), p. 270].

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