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ON ABSOLUTELY HENSTOCK INTEGRABLE FUNCTIONS

In this note we shall give an alternative definition of absolutely Henstock integrable functions.

A function f is said to be Henstock integrable (see [2]) on [a,b] if there exists a number A such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that whenever a division D given by

$$a = x_o < x_1 < \dots < x_n = b \quad \text{and} \quad \xi_1, \xi_2, \dots, \xi_n$$

satisfies $\xi_i - \delta(\xi_i) < x_{i-1} \leq \xi_i \leq x_i < \xi_i + \delta(\xi_i)$ for $i = 1,2,\dots,n$ we have

$$\left|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A\right| < \varepsilon,$$

or alternatively,

$$\left|\sum f(\xi)(v-u) - A\right| < \varepsilon$$

where [u,v] denotes a typical interval in D with $\xi \in [u,v]$

$\subset (\xi-\delta(\xi), \xi + \delta(\xi))$. A function f is said to be absolutely Henstock integrable if both f and $|f|$ are Henstock integrable.

The first author defines the following RL integral. A non-negative function f is said to be RL integrable on [a,b] if there exists a number A such that for every $\varepsilon > 0$ there exist an open set G and a constant $\delta > 0$ such that $|G| < \varepsilon$ and that for every division

$$a = x_o < x_1 < \dots < x_n = b$$

with $x_i - x_{i-1} < \delta$ for $i = 1,2,\dots,n$, and for any $\xi_i \in [x_{i-1}, x_i] - G$ we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \varepsilon.$$

It is understood that the term $f(\xi_i)(x_i - x_{i-1})$ is not included when $[x_{i-1}, x_i] - G$ is empty. Alternatively, we write

$$\left| \sum_{\xi \notin G} f(\xi)(v-u) - A \right| < \varepsilon.$$

As usual, we define A to be the integral of f on [a,b]. In general, a function f is RL integrable on [a,b] if both $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$ are RL integrable on [a,b], and we define

$$\int_a^b f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx.$$

We shall prove that the RL integral is equivalent to the absolute Henstock integral. Since the latter is uniquely defined, so is the former.

First we prove the equivalence for bounded functions.

THEOREM 1. Let f be bounded on [a,b]. Then f is RL integrable on [a,b] if and only if f is absolutely Henstock integrable on [a,b].

Proof. We may assume that f is nonnegative. Suppose f is RL integrable on [a,b]. Then given $\varepsilon > 0$ there exist an open set G and a constant $\delta > 0$ such that $|G| < \varepsilon$ and the rest of the condition holds. Define $\delta(\xi) = \delta$ when $\xi \notin G$ and $(\xi - \delta(\xi), \ \xi + \delta(\xi)) \subset G$ when $\xi \in G$. Then we see that f is Henstock integrable on [a,b].

Conversely, suppose f is nonnegative and Henstock integrable on [a,b]. Then we may choose a continuous function g (see, for example, [1] page 191) and an open set G such that $|G| < \varepsilon$ and

$$|f(x) - g(x)| < \varepsilon \quad \text{for } x \in [a,b] - G,$$

$$\left|A - \int_a^b g(x)dx\right| < \varepsilon$$

where A is the Henstock integral of f on [a,b].

Since g is continuous on [a,b], it is Henstock integrable on [a,b] with $\delta(\xi) = \delta > 0$, i.e. $\delta(\xi)$ is a constant function. Let $|g(x)| \leq M$ for all $x \in [a,b]$. Then for any division $D = \{[u,v]\}$ with $v-u < \delta$ we have

$$\begin{aligned}\left|\sum_{\xi \notin G} f(\xi)(v-u) - A\right| &\leq \left|\sum_{\xi \notin G}\{f(\xi)-g(\xi)\}(v-u)\right| + \left|\sum_{\xi \in G} g(\xi)(v-u)\right| \\ &+ \left|\sum g(\xi)(v-u) - \int_a^b g(x)dx\right| + \left|\int_a^b g(x)dx - A\right| \\ &< \varepsilon\,(b-a) + M\varepsilon + \varepsilon + \varepsilon.\end{aligned}$$

Hence f is RL integrable on [a,b].

THEOREM 2. A function f is RL integrable on [a,b] if and only if it is absolutely Henstock integrable on [a,b].

Proof. Again, we may assume that f is nonnegative. Suppose f is RL integrable on [a,b]. Let f^N denote the truncated function of f, i.e. $f^N(x) = f(x)$ when $f(x) \leq N$ and $f^N(x) = N$ when $f(x) > N$. We shall show that f^N is absolutely Henstock integrable on [a,b].

Since f is RL integrable on [a,b], given $\varepsilon > 0$ there exist an open set G and a constant $\delta > 0$ such that $|G| < \varepsilon$ and that for every division $D = \{[u,v]\}$ with $v - u < \delta$ we have

$$\left| \sum_{\xi \notin G} f(\xi)(v-u) - A \right| < \varepsilon.$$

Let $\omega(f; [u,v]-G)$ denote the oscillation of f over $[u,v] - G$ and $\omega = 0$ when $[u,v] - G$ is empty. Then it follows that

$$\begin{aligned} &\sum \omega(f^N; [u,v] - G)\big|[u,v] - G\big| + \omega(f^N; G)|G| \\ &\leq \sum \omega(f;[u,v] - G)(v-u) + 2N|G| \\ &< 2\varepsilon + 2N\varepsilon. \end{aligned}$$

That is, f^N is dominated above by a simple function and below by another simple function so that the difference of their integrals is small. Hence f^N is Lebesgue integrable and therefore absolutely Henstock integrable on [a,b].

Since $f^N(x)$ converges monotonely increasing to $f(x)$ as $N \to \infty$ and

$$\int_a^b f^N(x)dx \leq A \quad \text{for all } N,$$

then by the monotone convergence theorem for the Henstock integral (see [2]) f is absolutely Henstock integrable on [a,b].

Conversely, suppose f is nonnegative and Henstock integrable on [a,b]. Then the truncated function f^N is also Henstock integrable on [a,b]. Given

$\varepsilon > 0$, there are an integer N and $E_N = \{x;\ f(x) > N\}$ such that $N|E_N| < \varepsilon/2$ and

$$\left| \int_a^b f(x)dx - \int_a^b f^N(x)dx \right| < \varepsilon.$$

By Theorem 1, f^N is RL integrable on [a,b]. Let A_N denote the integral of f^N on [a,b]. Then there are an open set G_N and a constant $\delta > 0$ such that $N|G_N| < \varepsilon/2$ and that for every division $D = \{[u,v]\}$ with $v - u < \delta$ we have

$$\left| \sum_{\xi \notin G_N} f^N(\xi)(v-u) - A_N \right| < \varepsilon.$$

Write $G = E_N \cup G_N$ and note that $f^N(\xi) = f(\xi)$ when $\xi \notin G$. Then $|G| < \varepsilon$ and

$$\begin{aligned} \left| \sum_{\xi \notin G} f(\xi)(v-u) - A \right| &\le |A - A_N| \\ &\quad + \left| A_N - \sum_{\xi \notin G} f^N(\xi)(v-u) \right| \\ &< \varepsilon + \varepsilon + N|G| \\ &< 3\varepsilon. \end{aligned}$$

The above is so because if $[u,v] - G$ is non empty then $f^N(\xi)(v-u)$ is one of the terms in the sum over $\xi \notin G_N$, and if $[u,v]-G$ is empty, i.e. $[u,v] \subset G$, then $[u,v]-G_N$ may or may not be empty, the collection of all terms $f^N(\xi)(v-u)$, when $[u,v]-G_N$ is nonempty, is less than $N|G|$. Consequently, f is RL integrable on [a,b].

The following example shows that f being non-negative in the definition of the RL integral is essential. Let

$$\begin{aligned} f(x) &= 2n^2(n+1) \quad \text{when } x \in \left(\tfrac{1}{2}\left(\tfrac{1}{n+1} + \tfrac{1}{n}\right), \tfrac{1}{n}\right), \\ &= -\,2n^2(n+1) \quad \text{when } x \in \left(\tfrac{1}{n+1}, \tfrac{1}{2}\left(\tfrac{1}{n+1} + \tfrac{1}{n}\right)\right),\ n = 1,2,\ldots\ , \end{aligned}$$

and $f(x) = 0$ elsewhere in $[0,1]$. Then with $A = 0$ and for $\varepsilon > 0$ we can choose $G = [0,1/n)$ with $1/n<\varepsilon$ and $\delta>0$ such that for any division $D = \{[u,v]\}$ with $v-u<\delta$ we have

$$\left| \sum_{\xi \notin G} f(\xi)(v-u) - A \right| < \varepsilon.$$

In fact, given any real number A, it is easy to find an open set G (depending on A) such that the above inequality holds. Obviously, the function f is not even Henstock integrable on $[0,1]$.

The RL integral may also be regarded as a generalization of the dominated integral in [3]. For some other recent characterization of the Lebesgue integral, see [4,5,6].

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