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k-to-1 Functions on (0,1)

1. Introduction

If k is a positive integer, and if $f:X \longrightarrow Y$ is a function, then f is said to be <u>k-to-1</u> if, for every x in X, $f^{-1}f(x)$ contains exactly k elements. Jo Heath,[2], proved that a 2-to-1 function from either [0,1] or (0,1) into a Hausdorff space must be discontinuous at infinitely many points. Inspired by her paper, Kenneth Kellum and the author, [3], proved that, for each integer $k \geq 2$, a k-to-1 function from [0,1] onto itself must have infinitely many discontinuities. However, Harrold (see [1]) gives an example, which he attributes to G. E. Schweigert, of a 3-to-1 continuous function from [0,1] onto a circle (see Figure 1). Using this example, one can construct a k-to-1 continuous function from [0,1] onto a circle for any $k \geq 4$.

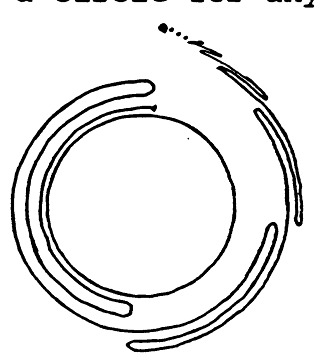

Figure 1.

This paper is an extension of [3] and we prove the following results:

(1) Let k be an integer ≥ 3. Consider a k-to-1 function f on (0,1) onto itself. If k is odd, then there is a continuous k-to-1 function f from (0,1) onto itself. If k is even, then f must have a discontinuities, and the number of discontinuity could be one.

(2) For every positive integer k, a k-to-1 function from (0,1) onto a simple closed curve S^1 must have infinitely many discontinuities.

(3) For every positive integer $k \geq 3$, the figure "8" is a k-to-1 continuous image of both [0,1] and (0,1).

The techniques used in this paper are very similar to those in [3].

2. k-to-1 functions from (0,1) onto itself.

Theorem 1. Let k be an integer ≥ 2. If f is a 2k-to-1 function from (0,1) onto itself, then f must have a discontinuity, and the least number of discontinuities of such a function f is one. There exists a (2k-1)-to-1 continuous function from (0,1) onto itself.

Proof: Suppose $f:(0,1) \longrightarrow (0,1)$ is a 2k-to-1 continuous surjection. Without loss of generality, we assume that

$$\lim_{x \to 0+} f(x) = 0, \text{ and } \lim_{x \to 1-} f(x) = 1,$$

for otherwise f has maximum or minimum values (see Lemma 2 of [3]). Let us denote the set $f^{-1}(1/2)$ by

$$\{x_1 < x_2 < \ldots\ldots < x_{2k}\},$$

where $0 < x_1 < x_{2k} < 1$. For each i = 1, 2, ..., 2k-1, $f[x_i,x_{i+1}]$ is an interval contained in (0,1/2] or [1/2,1). Then out of 2k-1 intervals $f[x_i,x_{i+1}]$, i = 1, 2, ..., 2k-1, at most k of them are contained in [1/2,1), and at most k of them are contained in (0,1/2]. Hence, without loss of generality, k intervals of the form $f[x_i,x_{i+1}]$ are contained in (0, 1/2], and let A be the index set $\{ i : f[x_i,x_{i+1}] \subset (0, 1/2] \}$. Then $|A| = k$, and let b be a number between $\max\{ \min f[x_i,x_{i+1}] : i \in A \}$ and 1/2. Then $f^{-1}(b) \cap [x_1,x_{2k}]$ contains at least 2k elements. But since the limit of f(x) as x approaches to 0 from the right is 0, and $f(x_1) = 1/2$, there also exists an element r in $(0,x_1)$ such that $f(r) = b$. Therefore, the set $f^{-1}(b)$ contains at least 2k+1 elements, which is a contradiction.

Now, the number of discontinuities of a 2k-to-1 function from (0,1) onto itself could be as small as one. Figures 2, 3, and 4 are graphs of 4-to-1, 6-to-1, and 8-to-1 functions, respectively, form (0,1) onto itself with one discontinuity. The general construction of a 2k-to-1 function from (0,1) onto itself is left to the reader.

Figure 5 is the graph of 3-to-1 continuous function from (0,1) onto itself. If one carefully replaces some line segments with "N's", the example can be made to be 5-to-1(see figure 6),

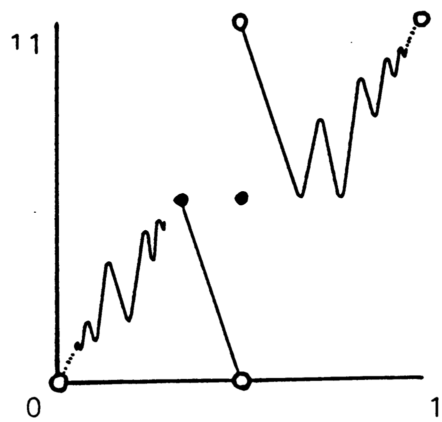

Figure 2

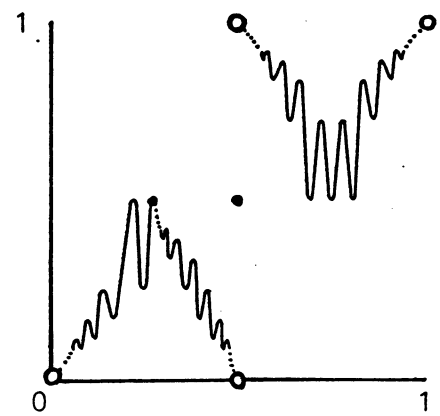

Figure 3

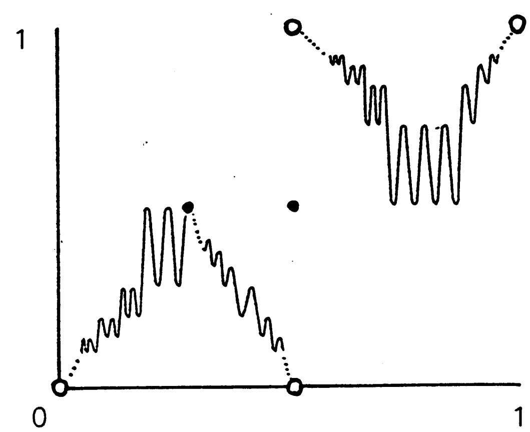

Figure 4

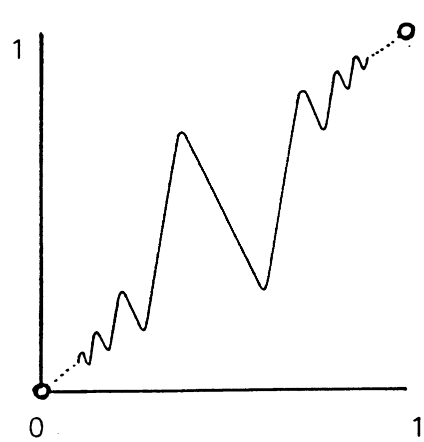

Figure 5

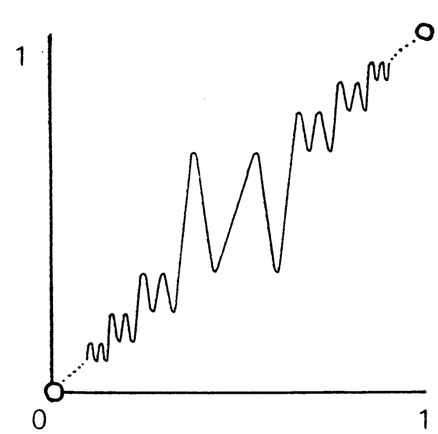

Figure 6

and this can be repeated to construct a (2k-1)-to-1 continuous function from (0,1) onto itself.

3. k-to-1 functions from (0,1) onto a circle.

We consider that a circle, S^1, to be a quotient space of the closed interval [0,1] obtained by identifying the two points 0 and 1 to a point, and denote the quotient map by $g:[0,1] \longrightarrow S^1$.

In order to prove the next theorem, we need one more lemma which is Theorem 2 in [3].

Lemma 3. Suppose f is a k-to-1 continuous function from an open set U of real numbers onto (0,1). Then the number of components of U is no more than k.

Indication of the proof: First, one must observe the following: If (a,b) is a component of U, then the both $\lim_{x\to a-} f(x)$ and $\lim_{x\to b+} f(x)$ are either 0 or 1.

Assume that U has more than k components, say $C_1, C_2, \ldots, C_k, C_{k+1}$ are components of U. From the above observation, $f(C_i)$ is either (0,1), (0,y], or [y,1) for some y in (0,1). Let P = {i: $f(C_i)$ = (0,y] for some y in (0,1)}, Q = {i: $f(C_i)$ = [y,1) for some y in (0,1)}, and R = { i : $f(C_i)$ = (0,1) }. Without loss of generality, assume that $|P| \geq |Q|$. Let a be a number between 0 and min{ $\max(f(C_i))$: $i \in P$ }. Then

$$|f^{-1}(a)| \geq 2|P| + |R| \geq |P| + |Q| + |R| = k+1.$$

But this is impossible since f is k-to-1.

Theorem 4: Let k be a positive integer. If $f:(0,1) \longrightarrow S^1$ is a k-to-1 surjection, then k has infinitely many discontinuities.

Proof: Assume f is only discontinuous at $d_1 < d_2 < \ldots < d_n$. Let us denote the set $f(\{d_1, d_2, \ldots, d_n\})$ by $\{s_1, s_2, \ldots, s_m\}$. Without loss of generality, assume that $0 = s_1 < s_2 < \ldots < s_m < 1$. Denote the set $f^{-1}(\{s_1, s_2, \ldots, s_m\})$ by $\{x_1, x_2, \ldots, x_{km}\}$, where $0 < x_1 < x_2 < \ldots < x_{km} < 1$. Let $s_{m+1} = 1$, $x_0 = 0$, $x_{km+1} = 1$. For each i = 1, 2, ..., m, write $f^{-1}((s_i,s_{i+1}))$ as U_i and let $f_i = f|_{U_i}:U_i \longrightarrow (s_i,s_{i+1})$. Then, for each i = 1, 2, ..., m, U_i is the finite union of open intervals of the form (x_j,x_{j+1}), say t_i many. Hence,

$$(*) \qquad t_1 + t_2 + \ldots + t_m = km + 1.$$

On the other hand, by Lemma 3, we must have, for each $i = 1, 2, \ldots, m$, t_i is less than or equal to k. Hence, we have

$$t_1 + t_2 + \ldots + t_m \leq km.$$

But this is a contradiction to (*).

4. A k-to-1 continuous image of both [0,1] and (0,1) for $k \geq 3$.

We will show that the figure "8" is a k-to-1 continuous image of both [0,1] and (0,1) for $k \geq 3$. More precisely, let X be the quotient space of [0,1] obtained by identifying three points 0, 1/2, and 1 to a point. Denote the quotient map by $h:[0,1] \longrightarrow X$. We will show a construction of k-to-1 continuous functions from both [0,1] and (0,1) onto X.

Let $f_3:[0,1] \longrightarrow [0,1]$ be the continuous function in Figure 7, $r:[0,1] \longrightarrow [0,1]$ the continuous function in Figure 8.

Suppose f_{3n} is defined for some $n \geq 1$ such that $f_{3n}(1)$ is either 0 or 1. Then we define $f_{3(n+1)}: [0,1] \longrightarrow [0,1]$ as follow:

$$f_{3(n+1)}(x) = \begin{cases} f_{3n}(2x) & \text{if } 0 \leq x \leq 1/2, \\ r(2x-1) & \text{if } 1/2 < x \leq 1 \text{ and } f_{3n}(1) = 0, \\ 1 - r(2x-1) & \text{if } 1/2 < x \leq 1 \text{ and } f_{3n}(1) = 1. \end{cases}$$

Let $f_7 = f_{3.2+1}: [0,1] \longrightarrow [0,1]$ be a function defined by;

$$f_7(x) = \begin{cases} f_3(3x) & \text{if } 0 \leq x \leq 1/3, \\ 1 - f_3(3x-1) & \text{if } 1/3 < x \leq 2/3, \\ 3x - 2 & \text{if } 2/3 < x \leq 1. \end{cases}$$

Suppose f_{3n+1} is defined for some $n \geq 2$ such that $f_{3n+1}(1)$ is either 0 or 1. Then we define $f_{3(n+1)+1}: [0,1] \longrightarrow [0,1]$ as follow:

$$f_{3(n+1)+1}(x) = \begin{cases} f_{3n+1}(2x) & \text{if } 0 \leq x \leq 1/2, \\ r(2x-1) & \text{if } 1/2 < x \leq 1 \text{ and } f_{3n+1}(1) = 0, \\ 1 - r(2x-1) & \text{if } 1/2 < x \leq 1 \text{ and } f_{3n+1}(1) = 1. \end{cases}$$

Let $s:[0,1) \longrightarrow (0,1]$ be the continuous function in Figure9.

Let $f_8 = f_{3.2+2}:[0,1] \longrightarrow [0,1]$ be a function defined by;

$$f_8(x) = \begin{cases} f_3(4x) & \text{if } 0 \leq x \leq 1/4, \\ s(4x-1) & \text{if } 1/4 < x \leq 1/2, \\ 2x - 1 & \text{if } 1/2 < x \leq 3/4, \\ -2x + 2 & \text{if } 3/4 < x \leq 1. \end{cases}$$

Suppose f_{3n+2} is defined for some $n \geq 2$. We define $f_{3(n+1)+2}:[0,1] \longrightarrow [0,1]$ as follow:

$$f_{3(n+1)+2}(x) = \begin{cases} f_{3n+2}(2x) & \text{if } 0 \le x \le 1/2, \\ r(2x-1) & \text{if } 1/2 < x \le 1 \text{ and } f_{3n+2}(1) = 0, \\ 1 - r(2x-1) & \text{if } 1/2 < x \le 1 \text{ and } f_{3n+2}(1) = 1. \end{cases}$$

Let f_4 and f_5 be the continuous function in Figure 10 and Figure 11, respectively. Then f_n is defined for all $n \ge 3$, and hf_n is a continuous n-to-1 function from [0,1] onto the figure "8".

Let $g_3:(0,1) \longrightarrow [0,1]$ be the continuous function in Figure 12. Let $t:[0,1) \longrightarrow [0,1)$ be the continuous function defined by $t(x) = s(x)$ for all x in [0,1).

Suppose g_{3n} is defined for some $n \ge 1$. Then we define $g_{3(n+1)}:(0,1) \longrightarrow [0,1]$ as follow:

$$g_{3(n+1)}(x) = \begin{cases} g_{3n}(2x) & \text{if } 0 < x < 1/2, \\ t(2x-1) & \text{if } 1/2 \le x < 1 \text{ and } \lim_{a\to 1-} g_{3n}(a) = 0, \\ 1 - t(2x-1) & \text{if } 1/2 \le x < 1 \text{ and } \lim_{a\to 1-} g_{3n}(a) = 1. \end{cases}$$

Let $g_7 = g_{3.2+1}:(0,1) \longrightarrow [0,1]$ be a function defined by;

$$g_7(x) = \begin{cases} g_3(3x) & \text{if } 0 < x < 1/3, \\ 1 - f_3(3x-1) & \text{if } 1/3 \le x \le 2/3, \\ 3x - 2 & \text{if } 2/3 < x < 1. \end{cases}$$

Suppose g_{3n+1} is defined for some $n \ge 2$. Then we define $g_{3(n+1)+1}:(0,1) \longrightarrow [0,1]$ as follow:

$$g_{3(n+1)+1}(x) = \begin{cases} g_{3n+1}(2x) & \text{if } 0 < x < 1/2, \\ t(2x-1) & \text{if } 1/2 \le x < 1 \text{ and } \lim_{a\to 1-} g_{3n+1}(a) = 0, \\ 1 - t(2x-1) & \text{if } 1/2 \le x < 1 \text{ and } \lim_{a\to 1-} g_{3n+1}(a) = 1. \end{cases}$$

Let $g_8 = g_{3.2+2}:(0,1) \longrightarrow [0,1]$ be a function defined by;

$$g_8(x) = \begin{cases} g_3(4x) & \text{if } 0 < x < 1/4, \\ s(4x-1) & \text{if } 1/4 \le x < 1/2, \\ 2x - 1 & \text{if } 1/2 \le x < 3/4, \\ -2x + 2 & \text{if } 3/4 \le x < 1. \end{cases}$$

Suppose g_{3n+2} is defined for some $n \ge 2$. Then we define $g_{3(n+1)+2}:(0,1) \longrightarrow [0,1]$ as follow:

$$g_{3(n+1)+2}(x) = \begin{cases} g_{3n+2}(2x) & \text{if } 0 < x < 1/2, \\ t(2x-1) & \text{if } 1/2 \le x < 1 \text{ and } \lim_{a\to 1-} g_{3n+1}(a) = 0, \\ 1 - t(2x-1) & \text{if } 1/2 \le x < 1 \text{ and } \lim_{a\to 1-} g_{3n+1}(a) = 1. \end{cases}$$

Let g_4 and g_5 be functions from (0,1) onto (0,1) in Figure 13 and Figure 14, respectively. Then $g_n:(0,1) \longrightarrow [0,1]$ is defined for

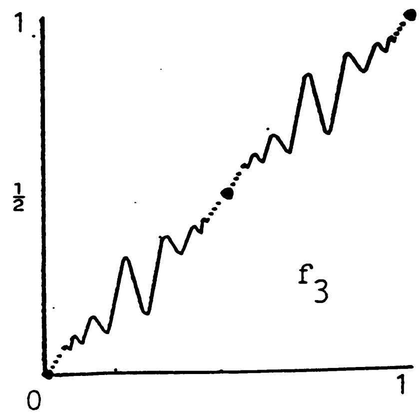

Figure 7.

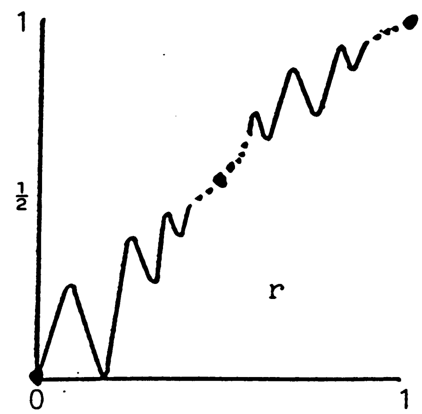

Figure 8.

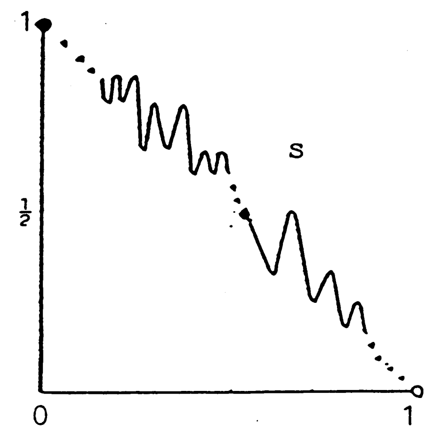

Figure 9.

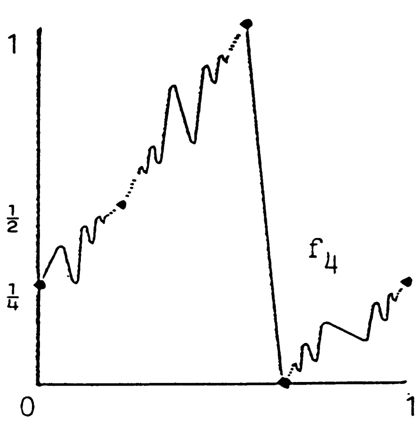

Figure 10.

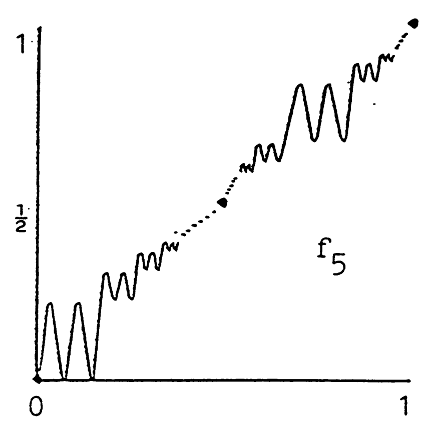

Figure 11.

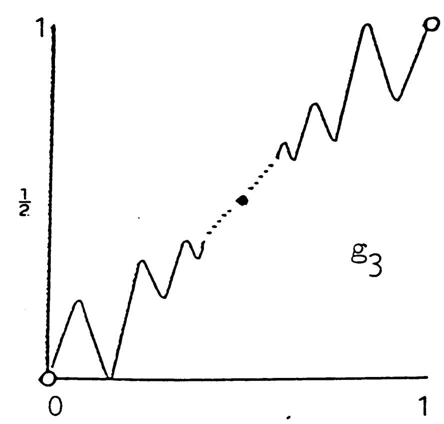

Figure 12.

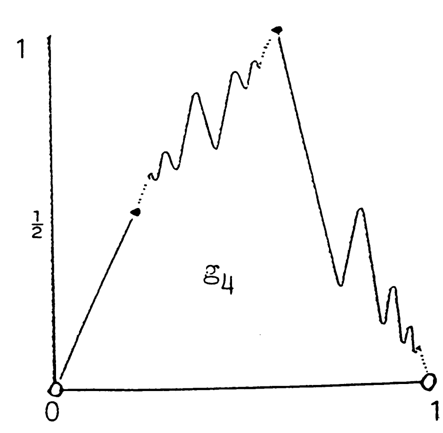

Figure 13.

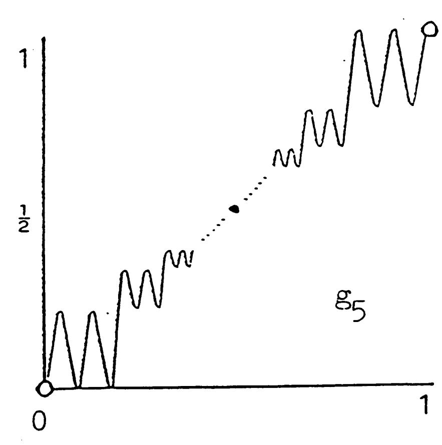

Figure 14.

every $n \geq 3$, and hg_n is a continuous n-to-1 function from (0,1) onto the figure "8".

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