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A COMPARISON OF THE JORDAN AND DINI TESTS

The purpose of this paper is to compare the Jordan and Dini tests for everywhere convergence of Fourier series. It is well known that these tests are classically noncomparable, but it is clear that in some sense the Dini test is much more powerful. We make this sense precise by associating with each test for everywhere convergence of Fourier series, an ordinal which we will call its <u>strength</u>. The strength is obtained by use of a natural measure of the complexity of functions with everywhere convergent Fourier series, called the Zalcwasser rank. We will show that the strength of the Jordan test is 3, while the strength of the Dini test is ω_1. A related question concerning the strength of the Young test is also mentioned.

We first discuss the Zalcwasser rank. This rank was first studied by Ajtai and Kechris [1], and was obtained by specialising a contruction of Zalcwasser [7] and (independently) Gillespie and Hurewicz [3]. Let T be the unit circle, and CS be the set of all everywhere convergent sequences of functions in C(T). Fix $\underline{f} = \langle f_n \rangle \in$ CS, and $\varepsilon > 0$. For each closed subset P of T, we define the <u>oscillation</u> $\omega(x;P)$ of $\underline{f}$ at x w.r.t. P by

$$\omega(x;P) = \inf_{\delta>0} \inf_{k\geq 1} \sup_{m,n\geq k} \{ |f_m(y) - f_n(y)| : y \in P \text{ and } |x-y| < \delta \}$$

we also define the derivative $(P)'_\varepsilon$ of P to be the set of all $x \in P$ with $\omega(x;P) \geq \varepsilon$. (Note that the derivative $(P)'_\varepsilon$ depends on both $\underline{f}$ and ε.) Since the f_n's are continuous it is easy to see that $(P)'_\varepsilon$ is closed. We may thus define a sequence $\langle Z^\alpha_\varepsilon \rangle$ by transfinite induction as follows. Put $Z^0_\varepsilon = T$, $Z^{\alpha+1}_\varepsilon = (Z^\alpha_\varepsilon)'_\varepsilon$, and $Z^\lambda_\varepsilon = \bigcap\{Z^\alpha_\varepsilon : \alpha < \lambda\}$ for λ a limit ordinal.

Now $(P)'_\varepsilon$ is nowhere dense in P. For suppose not, then there is an open interval I such that $\emptyset \neq I \cap P \subseteq (P)'_\varepsilon$. Let G_k be the set of all $x \in P$ such that there exist $m,n \geq k$ such that $|f_m(x) - f_n(x)| > \varepsilon/2$. Since the f_n's are continuous, it is clear that G_k is open. Also it is easy to see that G_k is dense in $I \cap P$, because $I \cap P \subseteq (P)'_\varepsilon$. Thus $\langle G_k \rangle$ is a sequence of dense, open sets in $I \cap P$. By the Baire category theorem it follows that $\bigcap\{G_k : k \geq 1\} \neq \emptyset$. Let $x_0 \in \bigcap\{G_k : k \geq 1\}$. Then from the definition of the G_k's we see that $\underline{f}$ diverges at x_0 - contradicting the fact that $\underline{f} \in CS$.

$\langle Z^\alpha_\varepsilon \rangle$ is therefore a strictly decreasing sequence of closed sets, and so by the Cantor - Baire stationary principle it must stabilise at $\emptyset$, for some countable ordinal. Define $\alpha(\varepsilon, \underline{f})$ to be the least ordinal at which the sequence stabilises.

Now observe from the definition of Z^α_ε, that $0 < \varepsilon_1 \leq \varepsilon_2$ implies $Z^\alpha_{\varepsilon_2} \subseteq Z^\alpha_{\varepsilon_1}$. So $\alpha(\varepsilon_2, \underline{f}) \leq \alpha(\varepsilon_1, \underline{f})$ for $0 < \varepsilon_1 \leq \varepsilon_2$. So $\sup\{\alpha(\varepsilon, \underline{f}) : \varepsilon > 0\} = \sup\{\alpha(1/n, \underline{f}) : n \geq 1\} < \omega_1$. For each $\underline{f} \in CS$, we now define the Zalcwasser rank of $\underline{f}$ by $|\underline{f}|_Z = \sup\{\alpha(\varepsilon, \underline{f}) : \varepsilon > 0\}$. The Zalcwasser rank is a natural measure of the complexity of the sequences in CS, in the sense that "nicely" convergent sequences have small rank, and vice versa.

In fact the sequences of rank 1 are precisely the uniformly convergent ones.

Indeed, suppose $\underline{f}$ is uniformly convergent. Then for each $\varepsilon > 0$, there is a $k \in \mathbb{N}$, such that $|f_m(x) - f_n(x)| < \varepsilon$, for all $m,n \geq k$ and $x \in T$. So $\omega(x;T) = 0$ for each $x \in T$. Thus $Z^1_\varepsilon = \emptyset$, for each $\varepsilon > 0$, and so $|f|_Z = 1$. Now suppose that $|f|_Z = 1$. Then for all $\varepsilon > 0$, $Z^1_\varepsilon = \emptyset$, and consequently $\omega(x;T) = 0$ for each $x \in T$. Now let $\varepsilon > 0$ be fixed. Then for each $x \in T$, there exists $\delta(x) > 0$ and $k(x) \in \mathbb{N}$, such that $|f_m(y) - f_n(y)| < \varepsilon$, for all $m,n \geq k(x)$ and y with $|x-y| < \delta(x)$. For each x, find such a $\delta(x)$ and $k(x)$. Since T is compact, there exists $x_1, \ldots, x_p \in T$, such that for any $x \in T$, there is an x_i such that $|x_i - x| < \delta(x_i)$. Let $k = \max\{k(x_1), \ldots, k(x_p)\}$. Then $|f_m(x) - f_n(x)| < \varepsilon$, for all $m,n \geq k$ and all $x \in T$. So $\underline{f}$ converges uniformly in T.

Let now CF be the set of all Lebesgue integrable functions on T, with everywhere convergent Fourier series. The Zalcwasser rank is defined on CF in the obvious way. For $f \in CF$, we define $|f|_Z$ to be the Zalcwasser rank of the sequence $< S_n(f) >$, of the partial sums of the Fourier series of f. Let $\mathcal{J}$ be a test for everywhere convergence of Fourier series. We define the strength of $\mathcal{J}$ by $S(\mathcal{J}) = \sup \{ |f|_Z + 1 : \mathcal{J}$ shows that the Fourier series of f converges everywhere $\}$. We will now compute the strengths of the Jordan and Dini tests. For convenience we give the tests below.

<u>Jordan test:</u> If f is a function of bounded variation on T, then the Fourier series of f converges everywhere.

Dini test: Let $\varphi(x,t) = f(x+t) + f(x-t) - 2f(x)$. If for each $x \in T$, $\int_0^\pi |\varphi(x,t) / t| \, dt < \infty$, then the Fourier series of f converges everywhere

We need the following result from [8 p. 61] .

Theorem A: Let f be a function of bounded variation on T . Then the Fourier series of f exhibits the Gibbs phenomenon at each essential discontinuity, and converges uniformly at each of the other points.

Propositon 1 : Let f be a function of bounded variation. Then $|f|_Z$ is 1, if f has no essential discontinuity, and is 2 otherwise.

Proof : If f has no essential discontinuity, then $S_n(f)$ is the same as the Fourier series of a continuous function of bounded variation on T. So $S_n(f)$ converges uniformly in T, and hence $|f|_Z = 1$. Now suppose f has an essential discontinuity. Then $S_n(f)$ is not uniformly convergent on T, and so $|f|_Z \geqslant 2$. We shall show that $|f|_Z \leqslant 2$. Let $\{x_n\}$ be the countably many points of essential discontinuity of f, and let d_n be their respective jumps. Then by theorem A, $\omega(x;T) = gd_n$ if $x = x_n$, and is 0 otherwise. (Here g is the Gibbs constant .) Since f is of bounded variation, there are finitely many x_n's , or there are infinitely many x_n's and $x_n \to 0$ as $n \to \infty$. So for a fixed $\varepsilon > 0$, $Z_\varepsilon^1 = \{x: \omega(x;T) \geqslant \varepsilon\}$ is a finite set. Since Z_ε^2 is nowhere dense in Z_ε^1 , it follows that $Z_\varepsilon^2 = \emptyset$, for each $\varepsilon > 0$. Thus $|f|_Z \leqslant 2$.

Corollary 2 : The strength of the Jordan test is 3.

We also need the following standard result in Descriptive set theory from [6 p. 213] .

Theorem B : Let X be a Polish space, A be a coanalytic subset of X, and $\varphi: A \to \omega_1$ be a coanalytic norm on A. If $B \subseteq A$ and φ is bounded in ω_1 on B, then B is an analytic subset of X.

Proposition 3 : Then strength of the Dini test is ω_1

Proof: Consider the Polish space C(T) and the set A = $C(T) \cap CF$. Ajtai and Kechris [1] showed that A is a coanalytic subset of C(T) and the Zalcwasser rank is a coanalytic norm on B. Let D(T) be the set of differentiable functions on T. Then $D(T) \subseteq A$, and D(T) is not an analytic subset of C(T) (see [5] or [4]) . So by theorem B , the Zalcwasser rank is unbounded in ω_1 on D(T). But the Dini test shows that each function in D(T) has an everywhere convergent Fourier series. Hence the strength of the Dini test is ω_1 .

It is clear that any test which is stronger than the Dini test will have strength ω_1 , so our notion of strength is not very useful for the Lebesgue test or the de la Valleé-Poussin test. The Young test is however non-comparable with the Dini test, and it is strictly stronger than the Jordan test (see[2 p. 263]) . So it would be of interest to find the strength of the Young test.

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