Real Analysis Exchange Vol. 12 Number 2 (1986-87)

Togo Nishiura, Department of Mathematics, Wayne State University, Detroit, MI 48202, Eldon E. Posey, and Jerry E. Vaughan, Department of Mathematics, University of North Carolina at Greensboro, Greensboro, NC 27412.

NOWHERE MONOTONE AND CANTOR FUNCTIONS

1. Introduction.

To each real-valued function f on an interval [a,b] we can associate an extended real-valued function (called the associate of f) by

$$\alpha[f](t) = \operatorname{Sup} f([a,t]),\ t \in [a,b].$$

Clearly, $\alpha[f]$ is a nondecreasing function, and when f is bounded, $\alpha[f]$ is real-valued and $\alpha[\alpha[f]] = \alpha[f]$. Other obvious properties are $\alpha[f \vee g] = \alpha[f] \vee \alpha[g]$ and $\alpha[\lambda f] = \lambda\alpha[f]$ for $\lambda \geqslant 0$. As the title of the paper indicates, we are interested in studying continuous functions f. The continuity of f implies the continuity of its associate $\alpha[f]$. In the sequel, we shall call $\alpha[f]$ trivial if $\alpha[f]$ is constant.

If f is the well-known continuous, nowhere differentiable function of Weierstrass (see [7], page 351) and the closed interval is [0,1], then $\alpha[-f]$, the associate of -f, is a nonconstant function whose intervals of constancy form a dense subset of [0,1]. (See [5] for a discussion of the denseness of the set of local extrema of nowhere differentiable functions. For differentiable functions, see the theorem of Zalcwasser concerning the denseness of the set of local extrema, [12] or [1], page 44.) The well-known Cantor ternary function F also has the property that the intervals of constancy of F form a dense subset of [0,1]. We make the following definition.

1.1 Definition. For a continuous function $f\colon [a,b] \to \mathbb{R}$, we define the set of constancy of f as

$K(f) = \{x \in [a,b] \mid f$ is constant on some neighborhood of $x\}$.

A continuous function f is called a Cantor function when its set of constancy $K(f)$ is a dense, proper subset of $[a,b]$.

(In [1], A. Bruckner refers to our Cantor functions by the name Cantor-like functions. We use the shorter name of Cantor functions in the present paper. The well-known Cantor function will be called the Cantor ternary function to avoid confusion.)

For a continuous function $f: [a,b] \to \mathbb{R}$, its set of constancy $K(f)$ is relatively open in $[a,b]$ and the set

$$P(f) = [a,b] \setminus K(f)$$

is dense-in-itself, hence perfect. Moreover, when f is a Cantor function, $P(f)$ is nonempty and nowhere dense in $[a,b]$.

In section 2 we investigate associates of continuous functions. The main results are: If f is a continuous, nowhere monotone function, then its associate $\alpha[f]$ is nondecreasing and is a Cantor function when $\alpha[f]$ is not trivial (Theorem 2.3). If f is in addition differentiable, then $\alpha[f]$ is differentiable except possibly at countably many points (Theorem 2.5 and Example 2.8). Conversely, if α_0 is a nondecreasing Cantor function, then there is a continuous, nowhere monotone function f_0 such that $\alpha[f_0] = \alpha_0$ (Theorem 2.9). If α_0 is in addition differentiable, then there is a differentiable f_0 which is nowhere monotone and $\alpha[f_0] = \alpha_0$.

Next, in section 3, we investigate the class of differentiable Cantor functions. To begin, we prove an analogue of a theorem of C. Goffman (Theorem 3.2). We also prove an analogue of a theorem of C. Weil (Theorem 3.8). These analogues are proved for the purpose of studying non-monotone Cantor functions. (Monotone Cantor functions were studied earlier in [11], [2] and [1].) Consider the following generalization of nowhere monotone functions.

1.2 Definition. a real-valued function f defined on a interval I in $\mathbb{R}$ is said to be nowhere monotone modulo intervals of constancy, if, whenever f is monotone on an open interval J contained in I, f is constant on J. (Here, we allow I to be unbounded.)

Our main theorem in this section is the existence of differentiable Cantor functions which are nowhere monotone modulo intervals of constancy (Theorem 3.4).

Finally, in section 4, we investigate the relationship between Cantor functions and the monotone-light factorization of continuous functions. Our main results of this section are characterizations of Cantor functions in general and Cantor functions which are nowhere monotone modulo intervals of constancy (Theorem 4.4). Also, a second existence theorem for differentiable Cantor functions which are nowhere monotone modulo intervals of constancy is proved (Theorem 4.5).

2. Associates of continuous functions.

In this section we present some properties of associates of continuous functions.

2.1 Proposition. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and $\alpha[f]$ be its associate. If $x \in P(\alpha[f])$, then $f(x) = \alpha[f](x)$.

Proof. We prove the contrapositive statement. Suppose $f(x) \neq \alpha[f](x)$. Then $f(x) < \alpha[f](x)$ and $x \neq a$. There is $\delta > 0$ such that $f(t) < \alpha[f](x)$ for $t \in (x-\delta, x+\delta) \cap [a,b]$, and $\delta < x-a$. The set $\{u \in [a,b] \mid f(u) = \alpha[f](x)$ and $u \leq x\}$ is not empty and is closed. Hence, it has a maximum element x_0. Clearly, $x_0 \leq x-\delta$ and $\alpha[f](x_0) = \alpha[f](t)$ for $t \in (x-\delta, x+\delta) \cap [a,b]$. We have shown that $x \in K(\alpha[f])$.

2.2 Corollary. If $P(\alpha[f])$ is somewhere dense in $[a,b]$, then f is strictly increasing on some open subinterval of $[a,b]$.

Observe that $K(f) \subset K(\alpha[f])$ and hence $P(f) \supset P(\alpha[f])$. One can easily construct an example to show the converse of Corollary 2.2 is not true.

2.3 Theorem. Let f be a continuous, nowhere monotone function on $[a,b]$. Then its associate $\alpha[f]$ is either trivial or a Cantor function. Indeed, it will be a Cantor function when and only when $f(a) < f(x)$ for some x in $[a,b]$.

We next investigate the differentiability properties of associates.

2.4 Proposition. Let $\alpha[f]$ be the associate for a continuous function f on $[a,b]$. Then the following three statements are true.

(1) $\alpha[f]$ is differentiable at each x in $K(\alpha[f])$ and its derivative is zero.

(2) If f is differentiable at an x in $P(\alpha[f])$ for which $(x-\delta, x) \cap P(\alpha[f]) \neq \emptyset$ for each $\delta > 0$, then $\alpha[f]$ is also differentiable at x and the derivative of $\alpha[f]$ at x is equal to $f'(x)$.

(3) If f is differentiable at an x which is a right end-point of a component of $K(\alpha[f])$ and is not b (and hence $x \in P(\alpha[f])$), then $\alpha[f]$ has a left derivative at x equal to zero and a right derivative at x equal to $f'(x)$.

Proof. The first assertion is obvious.

To prove the second assertion, let x be as in that assertion. By Proposition 2.1, we have $f(x) = \alpha[f](x)$. So, for $h < 0$, we have

$$\frac{f(x+h) - f(x)}{h} \geq \frac{\alpha[f](x+h) - \alpha[f](x)}{h}.$$

Hence,

$$f'(x) \geq \overline{\lim_{h\to 0-}} \frac{\alpha[f](x+h) - \alpha[f](x)}{h}.$$

Because $\alpha[f]$ is a nondecreasing function, there is a sequence h_n such that $h_n < 0$, $h_n \to 0$, $x+h_n \in P(\alpha[f])$ and

$$\lim_{n\to\infty} \frac{\alpha[f](x+h_n) - \alpha[f](x)}{h_n} = \underline{\lim_{h\to 0-}} \frac{\alpha[f](x+h) - \alpha[f](x)}{h}.$$

Since $f(x+h_n) = \alpha[f](x+h_n)$, we have that the left derivative of $\alpha[f]$ at x exists and equals $f'(x)$.

We next compute the right derivative of $\alpha[f]$ at x. Suppose $(x, x+\delta_0) \cap P(\alpha[f]) = \emptyset$ for some $\delta_0 > 0$. Then, $(x, x+\delta_0) \subset K(\alpha[f])$. Consequently, x is a point of local maximum for f. Since $f'(x)$ exists, $f'(x) = 0$. Clearly, the right derivative of $\alpha[f]$ at x is also zero when such a δ_0 exists. Finally, suppose $(x, x+\delta) \cap P(\alpha[f]) \neq \emptyset$ for every $\delta > 0$. Because $\alpha[f]$ is nondecreasing, there is a sequence h_n such that $h_n > 0$, $h_n \to 0$, $x+h_n \in P(\alpha[f])$, and

$$\lim_{n\to\infty} \frac{\alpha[f](x+h_n) - \alpha[f](x)}{h_n} = \overline{\lim_{h\to 0+}} \frac{\alpha[f](x+h) - \alpha[f](x)}{h}.$$

Moreover, for $h > 0$,

$$\frac{\alpha[f](x+h) - \alpha[f](x)}{h} \geq \frac{f(x+h) - f(x)}{h}.$$

Since $\alpha[f](x+h_n) = f(x+h_n)$, we have that the right derivative of $\alpha[f]$ at x exists and equals $f'(x)$.

We have proved that the right and left derivatives of $\alpha[f]$ at x exist and are equal to $f'(x)$. The second assertion is now proved.

As for the proof of the third assertion, we find that its two conclusions are consequences of parts of the proof of the second assertion. The details are left to the reader. Moreover, we remark that a detailed analysis of the Dini derivates of f and its associate $\alpha[f]$ can be made from the above calculations.

2.5 Theorem. If f is differentiable on [a,b], then its associate $\alpha[f]$ is differentiable at each x in [a,b] with the possible exception of the right end-points of the components of $K(\alpha[f])$. At these exceptional points x, the left derivative of $\alpha[f]$ at x is zero and the right derivative of $\alpha[f]$ at x is $f'(x)$, when $x \neq b$.

In the remainder of the paper, we will be using approximately continuous functions and the density topology [4]. For the convenience of the reader, we summarize the relevant facts about the density topology. The density topology consists of all Lebesgue measurable sets which have metric density equal to one at each of their points. Consequently, any set of measure zero is closed in the density topology. Clearly, the Euclidean topology is contained in the density topology. The set of approximately continuous functions is precisely the set of all real-valued functions which are continuous in the density topology. And, finally, the density topology is completely regular but not normal. In the sequel, we will use the modifiers "Euclidean" and "density topology" when we are dealing with topological conditions.

The next proposition will be used in Example 2.8 and Theorem 2.9, below.

2.6. Proposition. Let (a,b) be a bounded open interval. Then there is a differentiable function $f: \mathbb{R} \to [0,1]$ with the properties

1. f' is a bounded, approximately continuous function;
2. $\{x \mid f'(x) > 0\}$ and $\{x \mid f'(x) < 0\}$ are Euclidean dense subsets of (a,b);

and,

3. $f(x) = 0$ for $x \notin (a,b)$.

Proof. In [3], C. Goffman gives a simple construction of an approximately continuous function $\phi: \mathbb{R} \to [-1,1]$ such that $\{x \mid \phi(x) > 0\}$ and $\{x \mid \phi(x) < 0\}$ are Euclidean dense in $\mathbb{R}$. Let λ be a positive number and define $\phi_\lambda: \mathbb{R} \to \mathbb{R}$ by

$$\phi_\lambda(t) = t\,(1-t)\,[(\phi(t) \vee 0) + \lambda(\phi(t) \wedge 0)],$$

$t \in \mathbb{R}$. Then ϕ_λ is approximately continuous. For $x \in \mathbb{R}$, define

$$\Phi(x) = \int_0^x \phi_\lambda(t)\,dt.$$

By choosing λ appropriately, we may assume $\Phi(1) = 0$. Because $\Phi'(0) = 0 = \Phi'(1)$, there are two numbers x_0, x_1 in $[0,1]$ such that

$$\Phi(x_0) \leqslant \Phi(x) \leqslant \Phi(x_1)$$

for $x \in [0,1]$ and $\Phi'(x_0) = 0 = \Phi'(x_1)$. Using the restriction of Φ to the interval between x_0 and x_1, one easily constructs the required function.

2.7. Corollary. Let U be a bounded open subset of $\mathbb{R}$. Then there is a differentiable function $f: \mathbb{R} \to [0,1]$ with the properties

1. f' is a bounded, approximately continuous function;
2. $\{x \mid f'(x) > 0\}$ and $\{x \mid f'(x) < 0\}$ are Euclidean dense subsets of U;

and,

3. $f(x) = 0$ for $x \notin U$.

2.8 Example. There is a differentiable, nowhere monotone function $f:[0,1] \to \mathbb{R}$ such that its associate $\alpha[f]$ is not differentiable.

Proof. For the open intervals $(0,1/2)$ and $(1/2,1)$, let f_0 and f_1 be the corresponding functions given by Proposition 2.6. Let $x_1 \in (1/2,1)$ such that $x_1 \in P(\alpha[f_1])$ and $f_1'(x_1) > 0$. Next, let $y_0 = \max f_0(\mathbb{R})$. Then $y_0 > 0$ and $f_1(x_1) > 0$. Define $f\colon [0,1] \to \mathbb{R}$ by

$$f = \frac{f_1(x_1)}{y_0} f_0 + f_1 = \frac{f_1(x_1)}{y_0} f_0 \vee f_1.$$

Then,

$$\alpha[f] = \frac{f_1(x_1)}{y_0} \alpha[f_0] \vee \alpha[f_1].$$

Consequently, $\alpha[f]$ is not differentiable at x_1.

2.9. Theorem. Let $\alpha_0\colon [a,b] \to \mathbb{R}$ be a continuous, nondecreasing function. Then there is a continuous function $f_0\colon [a,b] \to \mathbb{R}$ such that $\alpha[f_0] = \alpha_0$ and f_0 is differentiable and nowhere monotone on $K(\alpha_0)$. If α_0 is a (differentiable) Cantor function then f_0 is a (differentiable) nowhere monotone function. Finally, f_0 can be chosen so that its total variation does not exceed $\alpha_0(b) - \alpha_0(a) + 1$.

Proof. With U equal to the Euclidean interior of $K(\alpha_0)$, let f be the function given by Corollary 2.7. Let $f_0 = \alpha_0 - f$. Clearly, f can be chosen so that $|f'(x)| \leq (b-a)^{-1}$ for all $x \in \mathbb{R}$. Then, $\int_a^b |f'(x)|\, dx$, the total variation of f on $[a,b]$, does not exceed 1.

3. Differentiable Cantor functions.

In the previous section, we referred to [3] in which C. Goffman showed a connection between the existence of a differentiable, nowhere monotone function and the density topology. In this section, we will extend Goffman's analysis to differentiable Cantor functions. (See [2] and [1] for earlier works on Cantor functions.) We will show a connection between the existence of a differentiable function which is nowhere monotone modulo intervals of constancy and the density topology. Furthermore, we will investigate the derivatives of differentiable Cantor functions in the spirit of C. Weil [8]. In particular, we will generalize Weil's theorem concerning typical functions in the space Δ_0 of bounded derivatives ϕ for which $\phi^{-1}(0)$ is dense. (See also [1], page 34, for a discussion of Weil's theorem.) To this end, we make the following definition.

3.1 Definition. Let U be any subset of $\mathbb{R}$. Then $b\Delta(U)$ will denote the set of bounded derivatives ϕ for which $\phi^{-1}(0) \supset U$.

Clearly, if $f:[a,b] \to \mathbb{R}$ is a differentiable Cantor function with bounded derivative, then $f' \in b\Delta(K(f))$. Conversely, if U is an open, dense subset of [a,b], and $\phi \in b\Delta(U)$, then the differentiable function f given by

$$f(x) = \int_a^x \phi(t)\,dt, \quad x \in [a,b],$$

is either a constant function or a differentiable Cantor function with $f' = \phi$.

For the convenience of the reader, we give a brief summary of facts about derivatives. Suppose $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function and let $Z(f') = \{x \mid f'(x) = 0\}$. The set $Z(f')$ is a G_δ-set and each component of $Z(f')$ is a closed subset of $\mathbb{R}$. Let Z_0 be the union of the nondegenerate components of $Z(f')$. Then, $Z(f') \setminus Z_0$ may have the cardinality of the continuum. Indeed, it was pointed out in [2] that $Z(f') \setminus Z_0$ has the cardinality of the continuum when f is a differentiable Cantor function. Finally, with $X = \mathbb{R} \setminus Z_0$, we infer from [10] (or [1]) that, if (u,v) is any open interval for which $(u,v) \cap X$ is not empty, then $(u,v) \cap X$ has positive measure.

Because our interest is in the set where a differentiable function is not locally constant, we are interested in sets X which satisfy the following two conditions:

<u>Completeness</u>: X is topologically complete; i.e., X is a G_δ-set in $\mathbb{R}$.

<u>Metric density</u>: If (u,v) is any open interval for which $(u,v) \cap X$ is not empty, then $(u,v) \cap X$ has positive measure (or, equivalently, the density topology interior of $(u,v) \cap X$ is not empty).

We are now ready to generalize a construction of C. Goffman [3].

3.2 <u>Theorem</u>. Let Z_0 be a subset of $\mathbb{R}$ such that $X = \mathbb{R} \setminus Z_0$ satisfies the completeness and metric density conditions above. Then there is a bounded, approximately continuous function $\phi: \mathbb{R} \to [-1,1]$ such that $\{x \mid \phi(x) > 0\}$ and $\{x \mid \phi(x) < 0\}$ are Euclidean dense subsets of X, $\phi^{-1}(0) \cap X$ is Euclidean dense in X, and $\phi^{-1}(0) \supset Z_0$.

<u>Proof</u>. Let V be the density topology interior of X. The metric density condition of X implies V is Euclidean dense in X and V is locally uncountable at each point of V. Let $D^0 = \{z_i \mid i = 1,2,\ldots\}$, $D^+ = \{p_i \mid i = 1,2,\ldots\}$ and $D^- = \{n_i \mid i = 1,2,\ldots\}$ be three disjoint, countable subsets of V which are Euclidean dense in V. Since D^0, D^+ and D^- are countable sets, they are closed in the density topology. For any i, we have that $F^+ = D^0 \cup D^- \cup (\mathbb{R} \setminus V)$ is a density topology closed set not containing p_i. By the complete regularity of the density topology, there

is an approximately continuous function $\phi_i^+ : \mathbb{R} \to [0,1]$ such that $\phi_i^+(p_i) = 1$ and $\phi_i^+(x) = 0$ for $x \in F^+$. Let $\phi^+(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \phi_i^+(x)$, $x \in \mathbb{R}$. Then ϕ^+ is an approximately continuous function on $\mathbb{R}$ into $[0,1]$ such that $\phi^+(x) > 0$ for $x \in D^+$ and $\phi^+(x) = 0$ for $x \in F^+$. Analogously, there is an approximately continuous function ϕ^- on $\mathbb{R}$ into $[0,1]$ such that $\phi^-(x) > 0$ for $x \in D^-$ and $\phi^-(x) = 0$ for $x \in F^-$, where $F^- = D^+ \cup D^0 \cup (\mathbb{R} \setminus V)$. Let $\phi = \phi^+ - \phi^-$. Then $\phi: \mathbb{R} \to [-1,1]$, and $\phi(x) = 0$ for $x \in D^0 \cup (\mathbb{R} \setminus V)$. Since $Z_0 = \mathbb{R} \setminus X \subset \mathbb{R} \setminus V$, the theorem is proved. (The above proof is only a slight modification of that given by C. Goffman. We have included it here for the sake of completeness.)

3.3 <u>Corollary</u>. Let Z_0 be a subset of $\mathbb{R}$ such that $X = \mathbb{R} \setminus Z_0$ satisfies the completeness and metric density conditions above. Let W be the Euclidean interior of Z_0. Then, there is a function $f: \mathbb{R} \to \mathbb{R}$ which is nowhere monotone modulo intervals of constancy, has $K(f) = W$, and has a bounded derivative.

<u>Proof</u>. Let $f(x) = \int_0^x \phi(t)\, dt$, $x \in \mathbb{R}$, where ϕ is the approximately continuous function of Theorem 3.2.

3.4 <u>Theorem</u>. Let P_0 be any nonempty, nowhere dense, perfect subset of $[0,1]$. Then there is a Cantor function f of bounded variation on $[0,1]$ such that f is nowhere monotone modulo intervals of constancy and $P(f) = P_0$. Moreover, if P_0 satisfies the metric density condition above, then f may be chosen to have a bounded derivative also.

<u>Proof</u>. Let P_1 be a nonempty, nowhere dense, perfect subset of $[0,1]$ such

that P_1 satisfies the completeness and metric density conditions above. (Such a set P_1 is easily constructed.) We may assume that $0 \in P_1$ when $0 \in P_0$ and $1 \in P_1$ when $1 \in P_0$. Let $h: [0,1] \to [0,1]$ be a homeomorphism such that $P_1 = h(P_0)$. Corollary 3.3 provides a Cantor function $\hat{f}: [0,1] \to \mathbb{R}$ such that $P(\hat{f}) = P_1$, $\hat{f}$ has bounded derivative, and $\hat{f}$ is nowhere monotone modulo intervals of constancy. Let $f = \hat{f}oh$. Since h is a homeomorphism, we have that f is of bounded variation, $P(f) = P_0$ and f is nowhere monotone modulo intervals of constancy.

3.5 <u>Remark</u>. Monotone, differentiable Cantor functions have been constructed earlier in [1], page 35, and in [11]. Clearly, they are easily constructed by using $\phi \vee 0$, where ϕ is the approximately continuous function from Theorem 3.2. The above Theorem 3.4, however, demonstrates a new and different kind of pathology for Cantor functions.

We next consider an extension of a theorem of C. Weil [8]. The following is an elementary observation.

3.6. <u>Proposition</u>: Let X be a G_δ subset of a complete metric space Y and let F be any nonempty set of bounded, real-valued functions f of Baire class one with $f^{-1}(0) \cap X$ dense in X. Then, the uniform closure of the vector space spanned by F is also a set of bounded, real-valued functions f of Baire class one with $f^{-1}(0) \cap X$ dense in X.

<u>Proof</u>. The proof is a simple application of the Baire Category Theorem. The important fact is that $f^{-1}(0)$ is a G_δ-set when f is of Baire class one.

3.7 Definition. Let Z_0 be a subset of $\mathbb{R}$ and denote $\mathbb{R} \setminus Z_0$ by X. Then, $\Delta_0(Z_0)$ is the subset of $b\Delta(Z_0)$ which consists of those bounded derivatives ϕ with the property that $\phi^{-1}(0) \cap X$ is Euclidean dense in X.

When $Z_0 = \emptyset$, $\Delta_0(Z_0)$ is the class Δ_0 defined by C. Weil [8]. When Z_0 is an F_σ-set, we have from Proposition 3.6 that $\Delta_0(Z_0)$ is a complete metric space with the uniform metric.

3.8 Theorem. Let Z_0 be a subset of $\mathbb{R}$ such that $X = \mathbb{R} \setminus Z_0$ satisfies the completeness and metric density conditions above. Then, the set of functions $\phi \in \Delta_0(Z_0)$ which are positive on one Euclidean dense subset of X and negative on another Euclidean dense subset of X form a residual subset of $\Delta_0(Z_0)$.

Proof. Let $\langle I_n \rangle$ be an enumeration of those open intervals I with rational end-points and $I \cap X$ not empty. For each n, let

$$E_n = \{\phi \in \Delta_0(Z_0) \mid \phi(x) \geq 0 \text{ for all } x \in I_n\}$$

and

$$F_n = \{\phi \in \Delta_0(Z_0) \mid \phi(x) \leq 0 \text{ for all } x \in I_n\}.$$

Clearly, each E_n and F_n are closed in $\Delta_0(Z_0)$. We will show that E_n is nowhere dense in $\Delta_0(Z_0)$.

Let $\phi \in E_n$ and $\delta > 0$. We assert that there is a point x_0 in the density topology interior of $I_n \cap X$ for which $\phi(x_0) < \delta$. To see this, consider two cases. First, suppose $\phi^{-1}(0) \supset I_n$. Then, the metric density condition satisfied by X implies $I_n \cap X$ has a nonempty density topology interior. Let x_0 be any point in this nonempty set. Second, suppose $I_n \setminus \phi^{-1}(0) \neq \emptyset$. Then, for $x_1 \in I_n \setminus \phi^{-1}(0)$, we have $\phi(x_1) > 0$. Because $\phi^{-1}(0) \cap I_n \neq \emptyset$, we have from the Darboux property of ϕ that the set $A_\delta = \{x \in I_n \mid 0 < \phi(x) < \delta\}$ is not empty. From Theorem 2.1 on page 87 of [1], we have that A_δ has positive measure. Since $\phi \in \Delta_0(Z_0) \subset b\Delta(Z_0)$, we also have $A_\delta \subset I_n \cap X$. Let x_0 be any point in the density topology interior of A_δ. Then $\phi(x_0) < \delta$. This completes the proof of our assertion. Next, let D be a countable, Euclidean dense subset of $(I_n \cap X) \setminus \{x_0\}$ and denote by W the density topology interior of $(I_n \cap X) \setminus D$. Since the density topology is completely regular, there is an approximately continuous function $\hat{\phi}: \mathbb{R} \to [0,1]$ such that $\hat{\phi}(x_0) = 1$ and $\hat{\phi}(x) = 0$ for $x \notin W$. One easily verifies that $\hat{\phi} \in \Delta_0(Z_0)$. With $\hat{\delta}$ such that $\phi(x_0) < \hat{\delta} < \delta$, we let $\phi_\delta = \phi - \hat{\delta}\,\hat{\phi}$. Then $\phi_\delta \in \Delta_0(Z_0)$ and the uniform distance between ϕ and ϕ_δ is $\hat{\delta}$. To see that $\phi_\delta \notin E_n$, we observe that

$$\phi_\delta(x_0) = \phi(x_0) - \hat{\delta}\,\hat{\phi}(x_0) = \phi(x_0) - \hat{\delta} < 0.$$

This completes the proof of the nowhere denseness of E_n in $\Delta_0(Z_0)$.

Since $-\phi \in E_n$ if and only if $\phi \in F_n$, we have that F_n is also nowhere dense in $\Delta_0(Z_0)$. Clearly, $\Delta_0(Z_0) \setminus \bigcup_{n=1}^{\infty} (E_n \cup F_n)$ is precisely the set of ϕ in $\Delta_0(Z_0)$ for which $\{x \mid \phi(x) > 0\}$ and $\{x \mid \phi(x) < 0\}$ are both Euclidean dense in X. This subset of $\Delta_0(Z_0)$ is residual in $\Delta_0(Z_0)$. This completes the proof of the theorem. (The proof is essentially the one given in [8] and in [1], page 34. The major difference is the use of Theorem 2.1 on page 87 of [1] in the proof that E_n is nowhere dense in $\Delta_0(Z_0)$.)

3.9 Remark. Clearly, the above Theorem 3.8 is of interest only when $\Delta_0(Z_0)$ has a nonzero member. By Theorem 3.2, this occurs when $\mathbb{R} \setminus Z_0$ satisfies the completeness and metric density conditions above. Also, the above Theorem 3.8 yields, in a very imprecise sense, that a typical differentiable Cantor function is nowhere monotone modulo intervals of constancy. The imprecision is due to the lack of a metric which makes this set of functions into a complete metric space.

4. A Characterization of Cantor functions.

We begin this section with a brief summary of monotone-light factorizations of continuous mappings from analytic topology. (See [9] for a reference.)

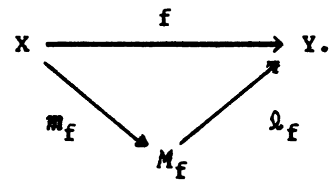
Let X and Y be compact metric spaces and $f: X \to Y$ be a continuous map. Then, we have the following definitions:

1. The map f is said to be monotone if $f^{-1}(y)$ is connected for each $y \in Y$.
2. The map f is said to be light if $f^{-1}(y)$ is totally disconnected for each $y \in Y$.

There is associated with the continuous map $f: X \to Y$ a decomposition of X into continua given by

$$\mathcal{D}_f = \langle C \mid C \text{ is a component of } f^{-1}(y) \text{ for some } y \in Y\rangle.$$

The decomposition $\mathcal{D}_f$ is upper-semicontinuous. Hence $\mathcal{M}_f = X/\mathcal{D}_f$, with the quotient topology, is metrizable. The natural projection map $m_f: X \to \mathcal{M}_f$ is a continuous, monotone map; and, the map $\ell_f: \mathcal{M}_f \to Y$ given by $\ell_f(\langle C\rangle) = f(C)$ $(m_f^{-1}(\langle C\rangle) = C)$ is a continuous, light map. Hence, f has a monotone-light factorization given by



When X is an interval and f is not constant, the space M_f is an interval and m_f is a nondecreasing, continuous function when M_f is appropriately parametrized. Let us summarize the above discussion into the following theorem.

4.1 <u>Theorem</u>. Suppose $f: [a,b] \to \mathbb{R}$ is a nonconstant, continuous function. Then, there is a nondecreasing, continuous, onto function $m_f: [a,b] \to [c,d]$ and there is a light function $\ell_f: [c,d] \to \mathbb{R}$ such that $f = \ell_f \circ m_f$. Moreover, $K(f) = K(m_f)$ and $K(\ell_f) = \emptyset$.

The following propositions are easily proved.

4.2 <u>Proposition</u>. Let $\alpha: [a,b] \to [c,d]$ be a continuous, nondecreasing, onto function and let $g: [c,d] \to \mathbb{R}$ be a continuous function. Then $K(g \circ \alpha) = K(\alpha) \cup \alpha^{-1}(K(g))$. Hence, if g is also a light function then $K(g \circ \alpha) = K(\alpha)$.

4.3 <u>Proposition</u>. Let $\alpha: [a,b] \to [c,d]$ be a continuous, nondecreasing, onto function and $g: [c,d] \to \mathbb{R}$ be a continuous function. Then $g \circ \alpha$ is of bounded variation when and only when g is of bounded variation.

The following theorem is now easily proved.

4.4 <u>Theorem</u>. Let $f: [a,b] \to \mathbb{R}$ be a continuous nonconstant function and f

$= \ell_f \circ m_f$ be its monotone-light factorization, with m_f: [a,b] → [c,d] nondecreasing and onto, and ℓ_f: [c,d] → $\mathbb{R}$. Then the following statements are true.

(1) The function f is nowhere monotone modulo intervals of constancy when and only when ℓ_f is nowhere monotone.

(2) The function f is a Cantor function when and only when m_f is a Cantor function.

The next existence theorem follows easily from the above.

4.5. Theorem. There is a Cantor function (respectively, differentiable Cantor function) which is nowhere monotone modulo intervals of constancy.

Proof. Let α:[0,1] → [0,1] be the Cantor Ternary function and g: [0,1] → $\mathbb{R}$ be a nowhere monotone, continuous function. Then $f = g \circ \alpha$ is an example of a Cantor function which is nowhere monotone modulo intervals of constancy. (For the differentiable case, we need only choose α to be a differentiable nondecreasing Cantor function and g to be differentiable as well as nowhere monotone.)

Some of the results in this paper were announced in the Abstracts, American Mathematical Society, [6].

Added in proof. It has come to the attention of the authors that Theorem 3.2 and its proof has appeared in the recently published book by J. Lukeš, J. Malý and L. Zajíček [13], page 261.

REFERENCES

1. A.M. Bruckner, Differentiation of Real Functions, Lecture Notes in Mathematics, Vol. 659, Springer-Verlag, Berlin, 1978.

2. A.M. Bruckner and J.L. Leonard, On differentiable functions having everywhere dense set of intervals of constancy, Canadian Math. Bull. 8 (1965), 73-76.

3. C. Goffman, Everywhere differentiable functions and the density topology, Proc. Amer. Math. Soc. 51 (1975), 250.

4. C. Goffman, C.J. Neugebauer and T. Nishiura, Density topology and approximate continuity, Duke Math. J. 28 (1961), 497-505.

5. E.E. Posey and J.E. Vaughan, Extrema and nowhere differentiable functions, to appear in Rocky Mountain J. Math.

6. E.E. Posey and J.E. Vaughan, Abstract #825-26-508, Nowhere monotone and Cantor functions, Abstracts Amer. Math. Soc. 7 (1986), 45.

7. E.C. Titchmarsh, The Theory of Functions, Oxford University Press, Oxford, 1939.

8. C. Weil, On nowhere monotone functions, Proc. Amer. Math. Soc. 56 (1976), 388-389.

9. G.T. Whyburn, Analytic Topology, American Mathematical Society Colloquium Publications, Volume 28, Providence, Rhode Island, 1942.

10. Z. Zahorski, Sur la première dérivée, Trans. Amer. Math. Soc. 69 (1950), 1-54.

11. Z. Zahorski, Über die Konstruktion einer differenzierbaren, monotonen, nicht konstanten Funktion, mit überall dichter Menge von Konstanzintervallen, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Vol. 30 (1937), 202-206.

12. Z. Zalcwasser, Sur les fonctions de Köpcke, Prace Math. Fiz. 35 (1927-28), 57-99.

13. J. Lukeš, J. Malý and L. Zajíček, Fine Topology Methods in Real Analysis and Potential Theory, Lecture Notes in Mathematics, Vol.1189, Springer-Verlag, Berlin, 1986.

Received September, 1986