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# A NOTE ON THE $\sigma$-IDEAL OF $\sigma$-POROUS SETS

In this note we shall show that $\gamma$-sets of reals are $\sigma$-porous and there exists a family of cardinality of the continuum of disjoint non-$\sigma$-porous perfect sets.

A family $\mathcal{G}$ of open subsets of $X$ is an $\omega$-cover of $X$ iff every finite subset of $X$ is contained in an element of $\mathcal{G}$. A space $X$ has the _$\gamma$-property_ ($X$ is a $\gamma$-set) iff for every $\omega$-cover $\mathcal{G}$ of $X$ there exists a family $\{D_m : m \in \omega\} \subseteq \mathcal{G}$ such that $X \subseteq \bigcup_k \bigcap_{m \geq k} D_m$.

For a subset of the reals we define the set

$$P(X) = \{x \in X : \limsup_{\varepsilon \to 0^+} l(X,x,\varepsilon)/\varepsilon > 0\}$$

where $l(X,x,\varepsilon)$ is the length of the longest subinterval of $(x-\varepsilon,x+\varepsilon)$ disjoint from $X$. A set $X \subseteq \mathbb{R}$ is called _porous_ if $P(X) = X$ and is called _$\sigma$-porous_ if it can be represented as countable union of porous sets.

**Theorem 1.** If $X \subseteq \mathbb{R}$ has the $\gamma$-property, then $X$ is $\sigma$-porous.

**Proof.** Let $X \subseteq \mathbb{R}$ be a $\gamma$-set. For every $0 < n < \omega$ and $A = \{x_1,x_2,\dots,x_n\} \subseteq X$ let $d_A = \min(\{|x_i-x_j| : i \neq j\} \cup \{\frac{1}{n}\})$. Define $U_A = \bigcup_{i=1}^{n} I_i$ where $I_i$ is an open interval such that $x_i \in I_i$ $|I_i| < \frac{1}{4} d_A$ and $\operatorname{dist}(I_i,I_j) > \frac{3}{4} d_A$ for $i \neq j$.

Let $\{y_n\}$ be a sequence of distinct elements of $X$. Define $\mathcal{G}_n = \{U_A - \{y_n\} : A \subseteq X$ has $n$ elements$\}$ and $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$. $\mathcal{G}$ is an $\omega$-cover of $X$. Thus there exists a family $\{D_m : m \in \omega\} \subseteq \mathcal{G}$ such that $X \subseteq \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} D_m$.

Since $y_n$ must be in all but finitely many $D_m$, we have that finitely many of $\{D_m : m \in \omega\}$ belong to $\mathcal{G}_n$.

We shall show that $X_k = \bigcap_{m=k}^{\infty} D_m$ is porous. Let $x \in X_k$ and $\varepsilon > 0$ and $n > \frac{1}{\varepsilon}$. Then there exist $m_0 > k$ and $n_0 > n$ such that $D_{m_0} \in \mathcal{S}_{n_0}$. There exists a set $A = \{x_1, x_2, \ldots, x_{n_0}\} \subseteq X$ such that $D_{m_0} = U_A - \{y_{n_0}\}$ and $x \in D_{m_0}$. So $D_{m_0} = \bigcup_{i=1}^{n_0} I_i - \{y_{n_0}\}$ where $I_i$ is an open interval such that $\text{dist}(I_i, I_j) > \frac{3}{4} d_A$ and $|I_i| < \frac{1}{4} d_A$. Assume that $x \in I_1$. Then $(x - \frac{3}{4} d_A, x + \frac{3}{4} d_A) \cap I_i = \emptyset$ for $i > 1$. Thus $((x - \frac{3}{4} d_A, x + \frac{3}{4} d_A) - I_1) \cap X_k = \emptyset$. Hence $(x - \frac{3}{4} d_A, x + \frac{3}{4} d_A) - I_1$ contains an interval longer than $\frac{1}{2} d_A$. So $l(X_k, x, d_A)/d_A > \frac{1}{2}$. Since $d_A < \varepsilon$, $\limsup_{\varepsilon \to 0^+} l(X_k, x, \varepsilon)/\varepsilon \geq \frac{1}{2}$.

It is not hard to see that the continuous image of a $\gamma$-set is a $\gamma$-set. F. Galvin and A.W. Miller [2] showed that assuming Martin's axiom there exists a $\gamma$-set of reals of cardinality of the continuum. They also stated that every set of reals of cardinality less than that of the continuum is a $\gamma$-set. This implies:

**Corollary 1.** Assume Martin's axiom. Every set of reals of cardinality less than that of the continuum is $\sigma$-porous.

**Corollary 2.** Assume Martin's axiom. There exists a set of reals $X$ of cardinality of the continuum such that every continuous image of $X$ is $\sigma$-porous.

**Remark.** A.W. Miller proved in [3] that it is consistent that for every $X \subseteq \mathbb{R}$ of cardinality of the continuum there exists a continuous function from $X$ onto $[0,1]$.

Assume that it is consistent that there exists a measurable cardinal. D.H. Fremlin and J. Jasiński [1] proved that it is consistent that there exists a set of reals $X$ of cardinality of the continuum such that every Borel image of $X$ has the $\gamma$-property.

**Corollary 3.** Assume that it is consistent that there exists a measurable cardinal. Then it is consistent that there exists a set of reals $X$ of cardinality of the continuum such that every Borel image of $X$ is $\sigma$-porous.

J. Tkadlec [5] showed that there exists an uncountable family of disjoint, non-$\sigma$-porous, perfect subsets of the reals. We shall prove a stronger theorem.

**Theorem 2.** There exists a family of cardinality of the continuum of disjoint, non-$\sigma$-porous, perfect subsets of reals.

**Proof.** By Theorem 1 of J. Tkadlec [5] there exists a non-$\sigma$-porous perfect subset of the reals $S$ such that $S - S$ is of the first category. ($S - S = \{s - s_1 : s, s_1 \in S\}$.) Let $G$ be a dense $G_\delta$ set such that $G \cap (S - S) = \emptyset$. Then $G \cup \{0\}$ is a dense $G_\delta$ set. By the result of J. Mycielski [4] there exists a perfect set $D$ such that $D - D \subset G \cup \{0\}$. So $(D - D) \cap (S - S) = \{0\}$. This implies that for every $t, w \in D$ such that $t \neq w$, $(t + S) \cap (w + S) = \emptyset$. Since $D$ is the cardinality of the continuum, we have the result.

## References


1. D.H. Fremlin and J. Jasiński, $G_\delta$ covers and large thin sets of reals, Journal of the London Math. Soc., to appear.

2. F. Galvin and A.W. Miller, $\gamma$-sets and other singular sets of real line, Topology Appl., 17 (1984), 145-155.

3. A.W. Miller, Mapping a set of reals onto the reals, The Journal of Symbolic Logic, Vol. 48 No. 3, Sept. 1983.

4. J. Mycielski, Independent sets in topological algebras, Fund. Math. 55 (1964), 139-147.

5. J. Tkadlec, Construction of Some Non-$\sigma$-porous Sets on the Real Line, Real Analysis Exchange, Vol. 9 No. 2, (1983-1984), 473-482.


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