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MONOTONICITY THEOREMS

In [2] Bruckner proved the following theorem:

Let f be a function satisfying the following conditions on an interval [a,b] : (i) f is a Darboux function in Baire's class one; (ii) f is VBG; (iii) f is increasing on each closed subinterval of [a,b] on which it is continuous and VB. Then f is continuous and nondecreasing on [a,b].

Bruckner obtained this result while answering affirmatively a problem presented by Zahorski in [24]. (This question was also answered independently by Swiatkowski in [23].)

In Chapter III we generalize Bruckner's theorem, but the proof of our theorem is shorter. We then give applications of this theorem which generalize consequences of Bruckner's theorem.

The following theorem of Banach ([21],p.286) is well known:

Any function which is continuous and satisfies Lusin's condition (N) on an interval, is derivable at every point of a set of positive measure.

Of course condition (N) implies condition T_2 and it is this fact that leads to the proof of Banach's theorem. In [9], Foran generalizes this result, showing that Banach's theorem remains true if condition (N) is replaced by Foran's condition (M).

An improvement of Foran's theorem is given in Chapter IV (Theorem 9), which is then used to prove a monotonicity theorem (Theorem 10) which generalizes the following result of Nina Bary ([1] or [21],p.286). (Condition (N) is replaced by condition (M).)

Every continuous function F which satisfies condition (N) and whose derivative is nonnegative at a.e. point x where F(x) is derivable, is monotone nondecreasing.

Further we give many applications of Theorem 10.

One of the most remarkable results of Chapter V is Corollary 7, which is a partial answer to the Open problem of this chapter.

CHAPTER I - PRELIMINARIES

For convenience, if P is a property for functions defined on a certain domain, we will also use P to denote the class of all functions having the property P. We denote by $\overline{A}$ the closure of the set A and by int(A) the interior of the set A. By B(F;X) we denote the graph of F on the set X. We denote by O(F;I) the oscillation of the function F on the interval I and by O(F;x) the oscillation of the function F at the point x. The set $X \subset R$ has a pair of isolated neighbours if there exist $x_1, x_2 \in R$ such that x_1 and x_2 are isolated in X and $(x_1,x_2) \cap X = \emptyset$. A property is said to hold n.e. (nearly everywhere) if it holds except on a countable set of points. Let $E_y = \{x : f(x) = y\}$. It is called a level set of the function f. Let $\mathcal{A}_1 \oplus \mathcal{A}_2$ (respectively $\mathcal{A}_1 \boxplus \mathcal{A}_2$) denote the linear space (resp. the semi-linear space) generated by the classes of functions $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{C}$ denote the class of all continuous functions and let D be the class of all Darboux functions on [0,1].

<u>Definition 1.</u>[10]. Given a natural number N and a set E, a function F is said to be B(N) on E if there is a number $M < +\infty$, such that for any sequence $\{I_k\}$ of nonoverlapping intervals with $I_k \cap E \neq \emptyset$, there exist intervals J_{kn}, n = 1,...,N, for which

$$B(F;E\cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} (I_k \times J_{kn}) \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < M.$$

Definition 2. [10]. Given a natural number N and a set E, a function F will be said to be A(N) on E if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $\{I_k\}$ are nonoverlapping intervals with $I_k \cap E \neq \emptyset$ and $\sum |I_k| < \delta$ then there exist intervals J_{kn}, $n=1,\dots,N$ such that $B(F; E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} (I_k \times J_{kn})$ and $\sum_k \sum_{n=1}^{N} |J_{kn}| < \varepsilon$.

Definition 3. [8]. Given a natural number N and a set E, a function F will be said to be E(N) on E if for every subset S of E, $|S| = 0$, and for any $\varepsilon > 0$ there exist rectangles $D_{kn} = I_k \times J_{kn}$, $n = 1,\dots,N$, with $\{I_k\}$ a sequence of nonoverlapping intervals, $I_k \cap S \neq \emptyset$ such that $B(F;S) \subset \bigcup_k \bigcup_{n=1}^{N} D_{kn}$ and $\sum_k \sum_{n=1}^{N} \operatorname{diam}(D_{kn}) < \varepsilon$.

Let $\mathfrak{F}$ (resp. $\mathfrak{B}$, $\mathscr{E}$) be the class of all functions F, defined on a closed interval I, for which there exist a sequence of sets E_n and natural numbers N_n such that $I = \bigcup E_n$ and F is $A(N_n)$ (resp. $B(N_n)$, $E(N_n)$) on E_n.

Let $\mathfrak{D}$ be an additive class of functions derivable in a sense which is compatible with the ordinary derivative $F'(x)$, i,e., $DF(x) = F'(x)$ at almost every point x where $F'(x)$ exists. Then $\mathfrak{F} \cap \mathfrak{D} \cap \mathscr{C}$ can be taken as a class of primitives and the $\mathfrak{F}\mathfrak{D}$ - integral (the Foran integral) can be defined by $\mathfrak{F}\mathfrak{D}\text{-}\int_a^b f(x)dx = F(b) - F(a)$, where $DF(x) = f(x)$ a.e. on $[a,b]$.

Definition 4. A function F fulfils Lusin's condition (N) on a set E if $|F(S)| = 0$ for every subset S of E for which $|S| = 0$.

Definition 5. A function $F:[0,1] \longrightarrow R$ is said to be B' on $E \subset [0,1]$ if there is a number $M < +\infty$ such that for any sequence $\{I_n\}$ of nonoverlapping intervals with $I_n \cap E \neq \emptyset$, there exists a sequence of closed sets K_n for which $B(F; E \cap \bigcup_n I_n) \subset \bigcup_n (I_n \times K_n)$ and $\sum_n |K_n| < M$.

Definition 6. [15]. A function $F:[0,1]\longrightarrow R$ is said to be $\overline{AC}$ on a set E if for every $\varepsilon>0$, there exists a $\delta>0$ such that $\sum(F(b_i)-F(a_i))<\varepsilon$ for each sequence of nonoverlapping intervals $[a_i,b_i]$, with endpoints in E and $\sum(b_i-a_i)<\delta$. Let $\underline{AC}=\{F: -F \in\overline{AC}\}$. Then $AC=\underline{AC}\cap\overline{AC}$.

Definition 7. A function F belongs to the class ACG (resp. VBG, $\mathfrak{B}'$) on a set E if $E=\bigcup E_n$ and F is AC (resp. VB, B') on each E_n. If condition AC is replaced by $\underline{AC}$ (resp. $\overline{AC}$) we obtain the class $\underline{ACG}$ (resp. $\overline{ACG}$). If the sets E_n are supposed to be closed we obtain the classes [ACG], [$\underline{ACG}$], [$\overline{ACG}$] and [VBG]. Clearly if $F|_E$ is B(N) then $F|_E\in B'$, hence $\mathfrak{B}\subset\mathfrak{B}'$.

Definition 8. A function $F:[0,1]\longrightarrow R$ satisfies condition $[\overline{M}]$ (resp. $[\overline{M}_*]$) on $E=\overline{E}\subset[0,1]$ if F is $\overline{AC}$ on each closed subset of E on which F is $VB\cap\mathcal{C}$ (resp. $VB_*\cap\mathcal{C}$). Let $[\underline{M}]=\{F : -F\in[\overline{M}]\}$; $[\underline{M}_*]=\{F : -F\in[\overline{M}_*]\}$; $[M]=[\overline{M}]\cap[\underline{M}]$; $[M_*]=[\overline{M}_*]\cap[\underline{M}_*]$. Note that F is $[M]$ (resp. $[M_*]$) on E if F is AC (resp. AC_*) on each closed subset of E on which F is $VB\cap\mathcal{C}$ (resp. $VB_*\cap\mathcal{C}$). (For the second part see Theorem 8.8, p.233 of [21].) Clearly $[M]\cap\mathcal{C}$ is identical with Foran's condition (M) (see [9]).

Definition 9. A function $F:[0,1]\longrightarrow R$ satisfies condition $[\overline{M}']$ if F is $\overline{AC}$ on each closed subinterval of [0,1] on which it is $VB\cap\mathcal{C}$. Let $[\underline{M}']=\{F : -F\in[\overline{M}']\}$; $[M']=[\overline{M}']\cap[\underline{M}']$.

Definition 10. A function $F:[0,1]\longrightarrow R$ satisfies Bruckner's condition B_i on [0,1] if F is increasing on each closed subinterval of [0,1] on which it is $VB\cap\mathcal{C}$. Let $B_d=\{F : -F\in B_i\}$.

Definition 11. [4]. The function F has the property D_d on [0,1] if $F([a,b])$ is everywhere dense on the closed interval with endpoints F(a) and F(b), for every subinterval [a,b] of [0,1].

Definition 12. [4]. The function $F\in D$ has the property D' on [0,1] if the values $y\in F([0,1])$ for which E_y is countably infinite and dense in the sense of order, form a null set. A function F has the property D" on $[0,1]$ if it has property D' on every interval $[a,b]\subset[0,1]$. (A set is dense in the sense of order if between every two points of E there is a point of E.)

Definition 13. The function $F\in D$ satisfies condition (D_r') on $[0,1]$ if the values $y\in F([0,1])$ for which E_y is countably infinite and for which E_y does not contain a pair of isolated neighbours form a null set. A function $F\in D$ has the property (D_r'') on $[0,1]$ if it is (D_r') on every subinterval of $[0,1]$.

Definition 14. [4]. The function F is [CG] (or B_1^*) on a set E if E can be expressed as the sum of a denumerable sequence of closed sets E_n over each of which F is continuous.

Definition 15. [16]. A function $F:[0,1]\longrightarrow R$ is uCM if F is increasing on the closed subinterval $[c,d]\subset[0,1]$ whenever it is so on the open interval (c,d). Let $lCM=\{F : -F\in uCM\}$ and let $CM = lCM\cap uCM$.

Definition 16. For a function $F:[0,1]\longrightarrow R$ we denote by (+) and (−) the following properties:

(+) for $0\leq a<b\leq 1$, if $F(a)<F(b)$ then $|F(P\cap[a,b])|\geq F(b)-F(a)$;

(−) for $0\leq a<b\leq 1$, if $F(a)>F(b)$ then $|F(N\cap[a,b])|\geq F(a)-F(b)$;

where $P=\{x : 0\leq F'(x)\leq +\infty\}$ and $N=\{x : 0\geq F'(x)\geq -\infty\}$.

Definition 17. [22]. Let $F:[0,1]\longrightarrow R$; $E^{+\infty}=\{x : F'(x)=+\infty\}$; $N^{+\infty}=\{F : |F(E^{+\infty})|=0\}$; $N^{-\infty}=\{F : -F\in N^{+\infty}\}$; $N^{\infty}=N^{-\infty}\cap N^{+\infty}$.

Remark 1. a) Conditions A(1) and AC are equivalent. Also conditions B(1) and VB are equivalent. (See [10].)

b) ACG is strictly contained in $\mathfrak{F}$ and VBG is strictly contained in $\mathfrak{B}$ (see [8]); $\mathfrak{F}\cap B_1^*$ is strictly contained in $\mathfrak{B}\cap B_1^*$ (see [10]); $\mathfrak{B}$ is strictly contained in $\mathfrak{B}'$ (see the function F constructed in the proof of Theorem 1,c) of [6]).

c) $\mathcal{F} \subset \mathcal{C} \subset (N) \subset [M]$ and all the inclusions are strict (see [6]).

d) $\mathcal{C} \oplus \mathcal{F} = \mathcal{C}$. (See the proof of Theorem 5.c) of [8].)

e) $[M] \subset [M_*]$ and $(N) \subset N^{\infty}$. (See [22],p.128.)

f) $\overline{AC} \subset VB \subset T_2$ for continuous functions on $[0,1]$.

g) $ACG \cap \mathcal{C} \subset [ACG] \subset ACG$.

h) $B_1^* \cap [VBG] \cap [M] = [\overline{ACG}]$ and $B_1^* \cap [ACG] = [ACG]$.

i) $DB_1 \oplus \mathcal{C} = DB_1$ on $[0,1]$. (See [3], Theorem 3.2,p.14.)

j) $(M) \oplus ACG = (M)$ for continuous functions on $[0,1]$. (See [6].)
In fact $[M] \boxplus (ACG \cap \mathcal{C}) = [M]$.

k) $(N) \subset T_2$ for measurable functions. (See [4], Theorem IV,p.473.)

l) $D \subset uCM$ on $[0,1]$. The converse is not true.

m) $D \oplus \mathcal{C} \subset D_d$ on $[0,1]$. (See [4], Theorem V,p.473.)

n) $D_d \oplus \mathcal{C} = D_d$ on $[0,1]$. (See [5].)

o) $(D_r') \subset D'$ and $(D_r'') \subset D''$ on $[0,1]$.

p) $VBG_* \cap \mathcal{C} \subset T_1 \subset T_2$. (See [21], Theorem 6.3,p.279.)

Lemma A. [6]. A Darboux function $F:[0,1] \longrightarrow R$ which is VB_* on a closed subset Q of $[0,1]$, is continuous on Q.

Theorem A. [9]. A continuous function $F:[0,1] \longrightarrow R$ satisfies (M) on $Q \subset [0,1]$ iff F is AC on any set $E \subset Q$ on which F is monotone.

Lemma B. [15]. Let $F:[0,1] \longrightarrow R$, $F \in AC$ on $[0,1]$ and $F'(x) \geqslant 0$ a.e. where $F'(x)$ exists. Then F is increasing on $[0,1]$.

CHAPTER II - RELATIONS BETWEEN SOME CLASSES OF FUNCTIONS

Theorem 1. a) There exists a function $F:[0,1] \longrightarrow [0,1]$, $F \in (D_d - D) \cap ACG$ such that $F = G + H$, where $H \in ACG \cap ((D_r'') - DB_1)$ and $G \in ACG \cap \mathcal{C}$.

b) D is strictly contained in D_d on $[0,1]$.

c) DB_1 is strictly contained in (D_r'') on $[0,1]$.

d) There is a function $f \in DB_2 \cap VBG$ such that $f \notin (D_r')$; There is a function $f_1 \in DB_2 \cap VBG \cap (D_r')$ such that $f_1 \notin (D_r'')$.

e) If $f \in (D_r'') \cap T_2$ on $[0,1]$, then f has properties (+) and (-).

f) If $f \in \mathfrak{B}$ on $[0,1]$ then f satisfies T_2 on $[0,1]$.

g) The class $[M]$ is strictly contained in $[M_*]$ on $[0,1]$.

h) The class $[M_*]$ is strictly contained in $[M']$ on $[0,1]$.

Corollary 1. $\mathscr{C} \subset DB_1 \subset (D_r'') \subset (D_r') \subset D \subset D_d$ and all the inclusions are strict.

Proof of Theorem 1. a) Let $I_{p,k} = [a_{p,k}, b_{p,k}]$, $k = 1,\dots,2^{p-1}$, be the closures of the intervals contiguous to C, from the p-th step in the Cantor ternary process. (C = the Cantor ternary set.) Let $E = [0,1] - \bigcup_{p,k} I_{p,k}$ and $d_{p,k} = (a_{p,k}+b_{p,k})/2$. Let $S_n = 1 + 2 + \dots + n$, $S_0 = 0$. Each point $x \in C$ is uniquely represented by $\sum c_i(x)/3^i$. Let $F(x) = 0$, $x \in E$ and $F(x) = i/(n+1)$, $x \in I_{S_n+i,k}$, $i = 1,2,\dots,n+1$. Let $G(x) = 0$, $x \in C$ and let $G(x) = 1/(n+1)$, $x = d_{S_n+i,k}$. Extending G linearly on each of the intervals $[a_{S_n+i,k}, d_{S_n+i,k}]$ and $[d_{S_n+i,k}, b_{S_n+i,k}]$, $i = 1,2,\dots,n+1$, we have G defined and continuous on $[0,1]$. Let $H(x) = F(x)-G(x)$. Then $H:[0,1] \longrightarrow [0,1]$ and $H(x) = 0$, $x \in E$; $H(x) = i/(n+1)$ if $x \in \{a_{S_n+i,k}, b_{S_n+i,k}\}$; $H(d_{S_n+i,k}) = (i-1)/(n+1)$; $H(x)$ is linear on each of the intervals $[a_{S_n+i,k}, d_{S_n+i,k}]$ and $[d_{S_n+i,k}, b_{S_n+i,k}]$, $i = 1,2,\dots,n+1$. Clearly $F,G,H \in ACG$ and $F \notin D$. Let $I = [a,b] \subset [0,1]$, $(a,b) \cap C \neq \emptyset$. Then there exists an interval $I_1 = [c,d]$, $c,d \in C$, $c = \sum_{i=1}^{S_n} c_i(c)/3^i$ and $d = c + \sum_{i=S_n+1}^{\infty} 2/3^i$, for some natural number n. I_1 contains 2^{j-1}

intervals contiguous to C from the step S_{n+j}, $j = 1,2,\ldots$, of the Cantor ternary process. We show that $F \in D_d$. Clearly $[0,1]$ contains the interval with endpoints $F(a)$ and $F(b)$, and $F(I) \supset \bigcup\{i/(n+j)\}$, $i = 1,2,\ldots,n+j$, $j = 1,2,\ldots$. Hence $\overline{F(I)} = [0,1]$ and $F \in D_d$. We show that $H \in (D''_r)$ on $[0,1]$. For each $i = 1,2,\ldots,n+1$, let $K(i)$ be a natural number such that $I_{S_n+i,K(i)} \subset I_1$. Then $H(I_{S_n+i,K(i)}) \subset [(i-1)/(n+1), i/(n+1)]$. Hence $H(I_1) = [0,1]$ and $H \in D$. Let $y \in [0,1]$ be an irrational number. Then $I_{S_n+i,K(i)} \cap E_y$ contains a pair of isolated neighbours, for some $i = 1,2,\ldots,n+1$, and $H \in (D''_r)$. We show that $H \notin DB_1$. Suppose on the contrary that $H \in DB_1$. Then by Remark 1,i), it follows that $H+G \in DB_1$. Contradiction.

b) The function F constructed in the proof of a) has the following properties: $F \notin D$ and $F \in D_d$.

c) The function H constructed in the proof of a) has the following properties: $H \in (D''_r)$ and $H \notin DB_1$. It remains to show that the class DB_1 is contained in (D''_r). Let $f: [0,1] \longrightarrow R$ be a DB_1 function. Let Y_2 be the set defined in [2](p.17), namely $Y_2 = \{y \in f([0,1])$: there is an $x \in E_y$ such that f attains a strict relative maximum or minimum at $x\}$. The set Y_2 is at most denumerable ([21],p.261). By the proof of Theorem 1 of [2](p.17) it follows that for every $y \in f([0,1]) - Y_2$, if E_y is denumerable then E_y contains a pair of isolated neighbours. Since $|Y_2| = 0$, $f \in (D'_r)$. Hence $DB_1 \subset (D''_r)$.

d) Let (a_n,b_n) be the intervals contiguous to C and let f be a function defined as follows: $f(a_n) = 1$, $n = 1,2,\ldots$; $f(x) = 0$, $x \in C$, $x \neq a_n$, $n = 1,2,\ldots$; f is linear and continuous on each $[a_n,b_n]$. By [2](pp.16-17), it follows that $f \notin (D'_r)$ and $f \in DB_2 \cap VBG$. Let $f_1(x) = f(x)$, $x \in [0,a_1) \cup [b_1,1]$; $f_1(a_1) = 0$; $f_1(x) = 1$, $x = (a_1+b_1)/2 = d_1$; $f(x)$ is linear and continuous on $[a_1,d_1]$ and $[d_1,b_1]$. Clearly $f_1 \notin (D'_r)$ on $[0,a_1]$ and $[b_1,1]$. Hence $f_1 \notin (D''_r)$.

For f the set E_y is countably infinite for each $y \in (0,1)$ and $E_y \cap (a_1,b_1)$ has a pair of isolated neighbours. Hence $f_1 \in (D_r')$.

e) In fact we prove more, namely: Suppose that $f \in (D_r') \cap T_2$ on $[0,1]$ and let $I_0 = [a,b] \subset [0,1]$. Then $P \cup N$ is nondenumerable, where $P = \{x : f'(x) \geq 0\}$, $N = \{x : f'(x) \leq 0\}$. If $f(a) < f(b)$ then $f(P)$ is measurable and $|f(P)| = |f([a,b])|$. If $f(a) > f(b)$ then $f(N)$ is measurable and $|f(N)| = |f([a,b])|$. The proof is analogous to that of Bruckner's Theorem 2 of [2](p.18). Let $Y_1 = \{y : E_y$ is nondenumerable$\}$. Let Y_2 be the set defined in the proof of c), and let $Y_3 = \{y : E_y$ is countably infinite and E_y does not contain a pair of isolated neighbours$\}$. Since $f \in T_2$, $|Y_1| = 0$. Y_2 is at most denumerable (see [21],p.261) and $|Y_3| = 0$ (since $f \in (D_r')$). Since $f \in D$, $f(I_0)$ is an interval. Hence $|f(I_0)| = |f(I_0) - (Y_1 \cup Y_2 \cup Y_3)|$. Suppose that $f(a) > f(b)$. For each $y \in f(I_0) - (Y_1 \cup Y_2 \cup Y_3)$ there is an isolated point x_y of E_y such that the upper bilateral derivative $\overline{f}'(x_y) \leq 0$. (If x_y is the only point of E_y then $\overline{f}'(x_y) \leq 0$ since $f(a) > f(b)$. If E_y is finite and contains more than one point then clearly E_y has a pair of isolated neighbours. If E_y is denumerable then E_y has a pair of isolated neighbours, since $f \in (D_r')$. Hence at one of these two points f' is nonpositive.) For each $y \in f(I_0) - (Y_1 \cup Y_2 \cup Y_3)$ select a point x_y such that $\overline{f}'(x_y) \leq 0$ and x_y is isolated in E_y. Let X be the set of points selected. Then $X = N \cup B$, where $B = \{x : -\infty < \overline{f}'(x) \leq 0$ and $\overline{f}'(x) \neq \underline{f}'(x)\}$, and $N \cap B = \emptyset$. By [21](p.270), $|f(B)| = 0$. Now $f(X) = f(N) \cup f(B)$ and $f(X) = f(I_0) - (Y_1 \cup Y_2 \cup Y_3)$. Hence $f(X)$ is measurable and $|f(X)| = |f(I_0)|$. It follows that $f(N)$ is measurable and $|f(N)| = |f(I_0)|$.

f) This follows by [10](p.360) and [11](p.35).

g) Let (a_{in}, b_{in}), $i = 1,2,\ldots,2^{n-1}$, be the intervals contiguous to C from the step n in the Cantor ternary process. (C = the Cantor ternary set .) Let f be a continuous function on $[0,1]$ defined as follows: $f(x) = 0$, $x \in C$; $f(x) = 1/n$, $x = (a_{in}+b_{in})/2 = d_{in}$; $f(x)$ is linear on $[a_{in},d_{in}]$ and $[d_{in},b_{in}]$, $i = 1,2,\ldots,2^{n-1}$. Clearly $f \in$ ACG on $[0,1]$. Let $g = f+\varphi$. (φ = the Cantor ternary function.) Clearly $g \in$ VBG on $[0,1]$. Since $g|_C$ is VB and $g|_C \notin$ AC ($\varphi(C) = [0,1]$), it follows that $g \notin (M)$. We show that $g \in [M_*]$ on $[0,1]$. Clearly $g'(x)$ does not exist (finite or infinite) for any point which is a right endpoint of some interval contiguous to C. Let $I_n = (a_n,b_n)$ be the intervals contiguous to C. Let $x \in C$, $x \neq 0$, $x \neq b_n$, $n = 1,2,\ldots$, and let $A(x) = \{n : c_n(x) = 2\}$. (Each $x \in C$ is uniquely represented by $\sum c_i(x)/3^i$.) Clearly $A(x)$ is a countable infinite set. For $n \in A(x)$ let $x_n' = \sum_{i=1}^{n-1} c_i(x)/3^i + \sum_{i=n}^{\infty} 2/3^i$, $x_n'' = \sum_{i=1}^{n-1} c_i(x)/3^i + \sum_{i=n+1}^{\infty} 2/3^i$ and $x_n''' = \sum_{i=1}^{n-1} c_i(x)/3^i + 2/3^n$. Clearly $x_n'' < x \leq x_n'$. Let $I_n(x) = (x_n'', x_n''')$. Clearly $I_n(x)$ is an interval contiguous to C from the step n in the Cantor ternary process. Let $d_n(x) = (x_n''+x_n''')/2$. Then $0 < x-d_n(x) < x_n'-x_n'' = 2/3^n$ and $g(x)-g(d_n(x)) = \varphi(x) - (\varphi(x_n''')+ 1/n) \leq \varphi(x_n') - \varphi(x_n''') - 1/n = 1/2^n - 1/n < 0$. Hence $|g(x)-g(d_n(x))/(x-d_n(x))| \geq (1/n - 1/2^n)/(2/3^n) \longrightarrow +\infty$, $n \to +\infty$. Therefore $\underline{f}_-(x) = -\infty$. But $\lim(g(y)-g(x))/(y-x) = \lim(\varphi(y) - \varphi(x))/(y-x) \geq 0$, $y \longrightarrow x$, $y \in C$. It follows that for each $x \in C$, $g'(x)$ does not exist (finite or infinite). Let E be a closed subset of $[0,1]$ such that $g|_E \in VB_*$. Then $g|_{E\cap C}$ is VB_* and $g'(x)$ does not exist for any point $x \in E \cap C$, $x \neq 0$. It follows by [21] (Theorem 7.2, p.230) that $|g(E\cap C)| = \Lambda(B(g;E\cap C)) = 0$. ($\Lambda(X)$ is the Hausdorff length of the set X.) Hence $g|_{E\cap C}$ satisfies Lusin's condition (N). By [21] (Theorem 6.7, p.227), $g|_E$ is AC. By [21] (Theorem 8.8, p.233), $g|_E \in AC_*$.

h) That $[M_*]$ is contained in $[M']$ follows easily by definitions. Let $I_n = (a_n,b_n)$ be the intervals contiguous to C. Let F be a continuous function on $[0,1]$ defined as follows: $F(x) = 0$, $x\in C$; $F(x) = h_n(x)$, $x\in I_n$, $h_n \in ACG_*-AC$, $|h_n(x)| < 1/2^n$. Let $G(x) = \varphi(x)+F(x)$. Clearly $G\in VBG_*$ on $[0,1]$ and $G_{|C} = \varphi_{|C} \in VB_*$. Since $G(C) = [0,1]$, $G\notin AC_*$ on C. Hence $G\notin [M_*]$. Let I be a subinterval of $[0,1]$, $int(I) \cap C \neq \emptyset$. Then for some n, $I\supset I_n$. Suppose on the contrary that $G_{|I}\in VB_*$. Then $G_{|\bar{I}_n} \in VB_*$. Since $G_{|I_n} \in ACG_*$, by [21](Theorem 6.7,p.227 and Theorem 8.8,p.233) it follows that $G_{|I_n} \in AC$. Contradiction. Hence if $G_{|I}\in VB_*$ then I is contained in some interval contiguous to C. Since $G\in ACG_*$ on each I_n it follows that $G\in AC_*$ on I and $G\in [M']$.

<u>Remark 2.</u> a) $(ACG_*\cap\mathcal{B})\oplus[M_*] = [M_*]$ <u>on</u> $[0,1]$, <u>but</u> $(ACG\cap\mathcal{B})\oplus [M_*] \neq [M_*]$ <u>on</u> $[0,1]$. (For the first part see Theorem 6.7,p.227 of [21]; The second part follows by the proof of Theorem 1,g).)
b) $AC\oplus[M'] = [M']$ <u>on</u> $[0,1]$ <u>but</u> $(ACG_*\cap\mathcal{B})\oplus[M'] \neq [M']$ <u>on</u> $[0,1]$. (See Theorem 6.7,p.227 of [21] and the proof of Theorem 1,h).)

<u>Remark 3.</u> The functions F and G constructed in the proof of Theorem 1,a) are identical to those of Example 1 and Example 4 of [4](pp.484-485).

<u>Theorem 2.</u> <u>A function</u> $F:[0,1]\longrightarrow R$ <u>belongs to</u> $D\cap(+)\cap N^{+\infty}$ <u>and</u> F' <u>is summable on</u> $P = \{x : F'(x)\geqslant 0\}$ <u>if and only if</u> $F\in \overline{AC}\cap\mathcal{B}$ <u>on</u> $[0,1]$.

<u>Proof.</u> Suppose that $F\in\overline{AC}\cap\mathcal{B}$. By Theorem 1,e) and Remark 1,f) it follows that $F\in(+)$. By [22](pp.136-137) it follows that $F\in N^{+\infty}$. The summability follows because $F\in VB$. Suppose that $F\in D\cap(+)\cap N^{+\infty}$ and F' is summable on P. Let $E^{+\infty} = \{x : F'(x) = +\infty\}$ and $E_+ = \{x : 0\leqslant F'(x) < +\infty\}$. Clearly $P = E^{+\infty}\cup E_+$. Let $g(x) = F'(x)$, $x\in E_+$; $g(x) = 0$, $x\notin E_+$ and let $G(x) = \int_0^x g(t)dt$. Since F' is summable on P it

follows that $G\in AC$ on $[0,1]$. Since $F\in N^{+\infty}\cap(+)$ it follows that $F(d)-F(c)\leqslant|F([c,d]\cap P)| = |F([c,d]\cap E_+)| \leqslant G(d)-G(c)$ for $0\leqslant c<d \leqslant 1$. (See [21], Theorem 6.5,p.227.) Let $\varepsilon>0$ and let δ be the number given by the fact that $G\in AC$ on $[0,1]$. Let $I_k=(a_k,b_k)$ be a sequence of nonoverlapping intervals such that $\sum|I_k|<\delta$. Then $\sum(F(b_k)-F(a_k))\leqslant \sum(G(b_k)-G(a_k))<\varepsilon$. Hence $F\in\overline{AC}$ on $[0,1]$. Since $\overline{AC}\subset VB$ and $F\in D$, it follows that F is continuous.

Corollary 2. a) $\overline{AC} = VB\cap N^{+\infty}$ for continuous functions on an interval. Hence $AC = VB\cap N^{\infty}$ for these functions.
b) Let $F:[0,1]\longrightarrow R$ be $DB_1\cap N^{+\infty}$. Then $F\in\overline{AC}\cap\mathcal{C}$ if and only if F' is summable over P.

Theorem 3. A function $f:[0,1]\longrightarrow R$ satisfies condition $[\overline{M}_*]$ (resp. $[\overline{M}]$) on a closed subset E of $[0,1]$ if and only if $f\in AC$ on any closed subset of E on which it is increasing and $VB_*\cap\mathcal{C}$ (resp. increasing and $\mathcal{C}$).

Proof. We prove only the part with $[\overline{M}_*]$. Suppose that f satisfies the second property and let $P=\overline{P}\subset E$ be such that $f|_P\in VB_*\cap\mathcal{C}$. Let $a=\inf(P)$, $b=\sup(P)$ and $F(x)=f(x)$, $x\in P$. Extending F linearly on each interval contiguous to P we have F defined, continuous and VB on $[a,b]$. Let $E_n=\{x\in[a,b] : (F(x+h)-F(x))/h>1, 0<|h|\leqslant 1/n\}$ and let $E_{in}=[i/n,(i+1)/n]\cap E_n$. By [21] (the proof of Theorem 10.1,pp.234-235), $F|_{E_{in}}$ is increasing and VB_*. Clearly $E_{+\infty}\subset(\bigcup_{i,n}E_{in})\cap P$, where $E_{+\infty}=\{x : F'(x)=+\infty\}$. The sets E_{in} may be supposed to be closed without loss of generality (see [21], Theorem 7.1,p.229). By hypothesis, $f|_{E_{in}\cap P}$ is AC. Since $|E_{+\infty}|=0$, it follows that $|F(E_{+\infty})|=0$. Hence $F|_{[a,b]}\in VB\cap N^{+\infty}=\overline{AC}$ (see Corollary 2). Hence $f\in[\overline{M}_*]$ on E. Conversely, let P be a closed subset of E such that $f|_P$ is increasing and $VB_*\cap\mathcal{C}$. Then by the definition of $[\overline{M}_*]$, $f|_P \in \overline{AC}$. Hence $f|_P \in AC$.

Theorem 4. Let $f:[0,1] \longrightarrow R$ be $[\overline{ACG}]$. Then $f \in [\overline{M}]$.

Proof. Let P be a closed subset of $[0,1]$ such that $f|_P$ is continuous and increasing. By hypothesis, there exists a sequence of closed sets P_n such that $f|_{P_n}$ is increasing and $\overline{AC}$. Hence $f|_{P_n}$ is AC. It follows that $f|_P$ is $VB \cap ACG = AC$. By Theorem 3 it follows that $f \in [\overline{M}]$.

Lemma 1. Let $f:[0,1] \longrightarrow R$ and let P be a closed subset of $[0,1]$. If $f|_P \in VB \cap \overline{ACG} \cap \mathscr{C}$ then $f|_P \in \overline{AC}$.

Proof. Let $F(x) = f(x)$, $x \in P \cup \{0,1\}$. Extending F linearly on each interval contiguous to $P \cup \{0,1\}$ we have F defined and $VB \cap \overline{ACG} \cap \mathscr{C}$ on $[0,1]$. By Theorem 4, $F \in VB \cap \mathscr{C} \cap [\overline{M}]$ on $[0,1]$. Hence $F \in \overline{AC}$ on $[0,1]$ and $f|_P \in \overline{AC}$.

Theorem 5. $[\overline{M}] \boxplus ([\overline{ACG}] \cap B_1^*) = [\overline{M}]$ on $[0,1]$.

Proof. Let $f \in [\overline{M}]$, $g \in [\overline{ACG}] \cap B_1^*$ and $h = f+g$. Let P be a closed subset of $[0,1]$ such that $h|_P \in VB \cap \mathscr{C}$. Then $f|_P \in [VBG] \cap B_1^*$, hence $f|_P \in [\overline{ACG}] \cap B_1^*$. By Lemma 1, $h|_P \in \overline{AC}$.

Lemma 2. Let $f:[0,1] \longrightarrow R$ and let P be a closed subset of $[0,1]$ such that $f|_P \in VB_*$, $a = \inf(P)$, $b = \sup(P)$. Let $F:[a,b] \longrightarrow R$, $F(x) = f(x)$, $x \in P$. On each interval $(c,d) \subset [a,b]$ contiguous to P, we define F such that its graph is the linear segment joining the points $(c,f(c))$ and $(d,f(d))$. Then there exists a set $N_0 \subset P$ such that $|f(N_0)| = |N_0| = 0$ and $f'(x) = F'(x)$ on $P - N_0$.

Proof. Let $E = \{x \in P : f'(x)$ does not exist finite or infinite$\}$; $E_1 = \{x \in P : F'(x)$ does not exist finite or infinite$\}$; $E_2 = \{x \in P : f'(x)$ and $F'(x)$ exist, $F'(x) \neq f'(x)\}$. Since $F(x) = f(x)$ on P, $F'(x) = f'(x)$ except perhaps at endpoints of intervals contiguous to P. Hence E_2 is a denumerable set. By [22] (Theorem 2, p.132) we have $|f(E)| = |E| = 0$. Since $F \in VB$ on $[a,b]$, $|F(E_1)| = |f(E_1)| = |E_1| = 0$. Clearly $|f(E_2)| = 0$. Let $N_0 = E \cup E_1 \cup E_2$. Then $|f(N_0)| = |N_0| = 0$ and $f'(x) = F'(x)$, $x \in P - N_0$.

Theorem 6. Let $f:[0,1] \longrightarrow R$, $f \in D$. Then $f \in N^{+\infty}$ if and only if $f \in [M_*]$ on $[0,1]$.

Proof. Suppose that $f \in N^{+\infty} \cap D$ and let P be a closed subset of $[0,1]$ such that $f|_P \in VB_* \cap \mathscr{C}$. Let $a = \inf(P)$, $b = \sup(P)$, $F(x) = f(x)$, $x \in P$. Extending F linearly on each interval contiguous to P we have F defined and $VB \cap \mathscr{C}$ on $[a,b]$. Let $E^{+\infty} = \{x \in P : f'(x) = +\infty\}$ and $E_1^{+\infty} = \{x \in P : F'(x) = +\infty\}$. By Lemma 2, $E_1^{+\infty} = (E_1^{+\infty} \cap N_0) \cup (E_1^{+\infty} \cap (P - N_0)) \subset N_0 \cup E^{+\infty}$, hence $|F(E_1^{+\infty})| = |f(E_1^{+\infty})| \leqslant |f(N_0)| + |f(E^{+\infty})| = 0$. Therefore $F \in VB \cap N^{+\infty} = \overline{AC}$ on $[a,b]$ (see Corollary 2). Hence $f|_P \in \overline{AC}$. Conversely, suppose that $f \in [\overline{M}_*] \cap D$. Let $E_{+\infty} = \{x : f'(x) = +\infty\}$ and $E_n = \{x \in [0,1] : (f(x+h) - f(x))/h > 1,\ 0 \leqslant |h| < 1/n\}$. Let $E_{in} = [i/n, (i+1)/n] \cap E_n$. By [21] (the proof of Theorem 10.1, pp.234–235), $f|_{E_{in}}$ is increasing and VB_* and $E_{+\infty} = \bigcup_{i,n} E_{in}$. The sets E_{in} may be supposed to be closed without loss of generality (see [21], Theorem 7.1, p.229). Since $f \in D$, $f|_{E_{in}} \in \mathscr{C}$ (see Lemma A). Since $f \in [\overline{M}_*]$, by Theorem 3, $f|_{E_{in}} \in AC$. Hence $|f(E_{+\infty})| = 0$ (since $|E_{+\infty}| = 0$).

Theorem 7. For functions defined on $[0,1]$ we have:
$[\overline{M}_*] \boxplus ([VBG_*] \cap [\overline{M}_*] \cap B_1^*) = [\overline{M}_*]$.

Proof. Let $f \in [\overline{M}_*]$, $g \in [VBG_*] \cap [\overline{M}_*] \cap B_1^* = [VBG_*] \cap \overline{ACG} \cap B_1^*$ and let $h = f+g$. Let P be a closed subset of $[0,1]$ such that $h|_P \in VB_* \cap \mathscr{C}$. Clearly $f|_P \in [VBG_*] \cap B_1^*$. By the definition of $[\overline{M}_*]$, $f|_P \in [\overline{ACG}]$. Hence $h|_P \in [\overline{ACG}]$. By Lemma 1, $h|_P \in \overline{AC}$.

Remark 4. Theorem 7 generalizes a result of [22] (Theorem 10, p.147).

CHAPTER III - AN EXTENSION OF BRUCKNER'S MONOTONICITY THEOREM. APPLICATIONS.

Theorem 8. Let $f:[0,1]\longrightarrow R$ be a function satisfying the following conditions on $[0,1]$: (i) $f\in D\cap(-)$; (ii) $f\in \mathcal{B}'$ on $H = \{x\in [0,1] : f$ is continuous at $x\}$; (iii) $f\in B_1$ on $U(f) = \text{int}(H)$. Then f is continuous and increasing on $[0,1]$.

Remark 5. Note that Bruckner's theorem follows from Theorem 8: $VBG\subset T_2$ (see [21],p.279); $DB_1\subset (D_r'')$ and $(D_r'')\cap T_2\subset (-)$ (see Theorem 1,c),e)); $VBG\subset \mathcal{B}'$ (see Remark 1,b)); it follows that f satisfies the conditions of Theorem 8.

Let (i') $f\in (D_r'')\cap T_2$; (i") $f\in DB_1\cap T_2$; (iii') $f\in [\underline{M}']$ on $U(f)$ and $f'(x)\geq 0$ a.e. where $f'(x)$ exists on $U(f)$. If in Theorem 8: a) condition (i) is replaced by (i') or (i"); b) condition (iii) is replaced by (iii'); c) condition (i) is replaced by (i') or (i") and condition (iii) is replaced by (iii'); then we obtain some additional monotonicity theorems. (Condition (i') implies (i) (see Theorem 1,e)). Condition (i") implies (i) (see Theorem 1,c),e)). Condition (iii') implies (iii): let $I\subset U(f)$ be a closed interval such that $f\in VB\cap\mathcal{C}$ on I. Since $f\in [\underline{M}']$ on $U(f)$, $f\in \underline{AC}$ on I. By Lemma B, f is increasing on I. Hence $f\in B_1$ on $U(f)$.)

Lemma 3. Let $F:[0,1]\longrightarrow R$ be a VBG_* function on a nondenumerable set $Q\subset [0,1]$. Then F is continuous n.e. on Q.

Proof. (The proof is similar to that of [3],pp.196-197). Since $F\in VBG_*$ on Q, it follows that there exists a sequence of sets Q_i such that $Q = \bigcup Q_i$ and $F|_{Q_i}$ is VB_*. By [21] (Theorem 7.1,p.229), $F|_{\overline{Q_i}}\in VB_*$. Let $E_n = \{x : O(F;x)\geq 1/n\}$. Then E_n is closed for each n. If $E_n\cap(\bigcup \overline{Q_i})$ is nondenumerable then there exists a natural number

i_0 such that $E_n \cap \overline{Q_{i_0}}$ is nondenumerable. Let P be a nonempty perfect subset of $E_n \cap \overline{Q_{i_0}}$. Clearly $F|_P \in VB_*$. Since $P \subset E_n$, $O(F;x) \geq 1/n$ for all $x \in P$. Thus the oscillation of F on any interval determined by two bilateral limit points of P is at least 1/n. Since P is perfect we can choose as many such intervals as we like, and we can make them pairwise disjoint. It follows that $F \notin VB_*$, a contradiction. Thus the set of points of discontinuity of F is at most denumerable.

Lemma 4. Let $f \in D \cap (-)$ on $[0,1]$ and let $H = \{x \in [0,1] : f$ is continuous at $x\}$. Then H is a G_δ-set, everywhere dense in $[0,1]$.

Proof. Let $J \subset [0,1]$ be an interval. If $f|_J$ is monotone then by the Darboux property, $f \in \mathcal{C}$. Hence $J \cap H \neq \emptyset$. If f is not monotone then there exist $x_1, x_2 \in J$, $x_1 < x_2$ such that $f(x_1) > f(x_2)$. Then by (-), $|f(N \cap [x_1,x_2])| \geq f(x_1) - f(x_2)$. Hence $N \cap [x_1,x_2]$ is nondenumerable and $f|_{N \cap [x_1,x_2]}$ is VBG_* (see [21], p.234). By Lemma 3, $[x_1,x_2]$ contains uncountably many points of continuity. Hence H is a G_δ-set, everywhere dense in $[0,1]$.

Lemma 5. Let $f \in D \cap (-)$ on $[0,1]$. If $f \in \mathcal{B}'$ on $H = \{x \in [0,1] :$ f is continuous at $x\}$ then there exists a sequence $\{I_n\}$ of intervals whose union is dense in $[0,1]$ and on each of which $f \in VB \cap \mathcal{C}$.

Proof. Since $f \in \mathcal{B}'$ there exists a finite or denumerable sequence of sets H_n such that $H = \bigcup H_n$ and $f \in B'$ on H_n. By Lemma 4, H is a G_δ-set, everywhere dense in $[0,1]$. By Baire's Category theorem there exist a positive integer p and an interval J such that $H \cap \operatorname{int}(J) \neq \emptyset$ and $J \cap H \subset \overline{H_p}$. We show that $f|_{J \cap H} \in B'$. Let $I \subset J$ be an interval and let $f(I \cap H_p) \subset K_p = \overline{K_p}$. Then $f(I \cap H) \subset K_p$. By definition it follows now that f is B' on $J \cap H$. We show that $f \in VB$ on $J = [a,b]$. Suppose on the contrary that $f \notin VB$ on J. Then there exists a division of J, namely $a = a_0 < a_1 < \dots < a_{n+1} = b$, such that

(1) $$\sum_{i=0}^{n} |f(a_{i+1})-f(a_i)| \geq 4\cdot M + |f(b) - f(a)|,$$

where M is the positive real number given by the fact that $f \in B'$ on $J \cap H$. Let $\mathcal{A} = \{i : f(a_{i+1}) < f(a_i)\}$. Since $\sum(f(a_{i+1})-f(a_i)) = f(b) - f(a)$, by (1) it follows that

(2) $$\sum_{i \in \mathcal{A}} (f(a_i)-f(a_{i+1})) \geq 2\cdot M.$$

Since $f \in (-)$, $f(a_i)-f(a_{i+1}) \leq |f(N \cap [a_i, a_{i+1}])|$ for each $i \in \mathcal{A}$. It is well known that the set $\{x : f'(x) = 0\}$ maps onto a set of measure 0 ([21], Theorem 4.5, p.271). It follows that $N_i = \{x \in [a_i, a_{i+1}] : -\infty \leq f'(x) < 0\}$ is nondenumerable for each $i \in \mathcal{A}$. Furthermore, $f \in VBG_*$ on N_i ([21], p.234). Let $N_i' = \{x \in N_i : f \text{ is continuous at } x\}$. Clearly $N_i' \subset H$. By Lemma 3, we have

(3) $$f(a_i)-f(a_{i+1}) \leq |f(N_i' \cap [a_i, a_{i+1}])| \leq |f(H \cap [a_i, a_{i+1}])| .$$

For each $i \in \mathcal{A}$ let K_i be closed sets such that $f(H \cap [a_i, a_{i+1}]) \subset K_i$. By (2) and (3), $\sum_{i \in \mathcal{A}} |K_i| \geq 2\cdot M$. Contradiction. Hence $f \in VB$ on J. Since $f \in D$ it follows that f is continuous on J. The argument we have just given applies equally well to any subinterval of $[0,1]$. The conclusion of our lemma follows by repeated application of this process.

<u>Proof of Theorem 8.</u> By (i), (ii) and Lemma 5, it follows that there exists a sequence of intervals $\{I_n\}$ whose union is dense in $[0,1]$ and on each of which $f \in VB \cap \mathcal{C}$. Let $[c_n, d_n] \subset I_n$. By (iii), f is nondecreasing on $[c_n, d_n]$. Since $[c_n, d_n]$ was an arbitrary subinterval of I_n, it follows that f is increasing on each I_n. The intervals I_n can be chosen to be maximal open intervals of monotonicity of f. We wish to show that in fact there exists only one such maximal interval, namely the interior of $[0,1]$. Suppose that there

is more than one such maximal interval and let $Q = [0,1] - (\bigcup I_n)$. The set Q is a nonempty perfect subset of $[0,1]$ ([2],pp.20-21). Let $H_1 = H \cap Q$. Then H_1 is a G_δ-set. We show that H_1 is everywhere dense in Q. Let J be an open subinterval of $[0,1]$ containing points of Q. Let $x_0 \in Q \cap J$. Since $x_0 \in Q$, f cannot be nondecreasing on all of J. Thus J contains points z_1 and z_2, $z_1 < z_2$ such that $f(z_1) > f(z_2)$. Let $N = \{x : -\infty \leq f'(x) \leq 0\} \cap [z_1, z_2]$. The set $\{x : f'(x) = 0\}$ maps onto a set of measure zero, from which it follows (since $f \in$ (-)) that $N' = \{x \in [z_1, z_2] : -\infty \leq f'(x) < 0\}$ is nondenumerable. By [21] (p.234), f is VBG_* on N'. Let $N'' = \{x \in N' : f$ is continuous at $x\}$. Clearly $N'' \subset H_1$. By Lemma 3 and (-) we have $f(z_1) - f(z_2) < |f(N'')| \leq |f(H_1 \cap [z_1, z_2])|$. Hence $[z_1, z_2] \subset J$ contains an uncountable set of points of continuity. Hence H_1 is everywhere dense in Q and a G_δ-set. Now the proof continues analogously to that of Lemma 5, if the set H (in the proof of Lemma 5) is replaced by H_1. Therefore we obtain that $f \in VB$ on J. Since $f \in D$, f is continuous on J. Hence $J \subset U(f)$. Let $(c,d) \subset J$, $c,d \in Q$. By (iii), f is increasing on $[c,d]$, a contradiction, since $[c,d]$ contains infinitely many points of Q.

Remark 6. If $f \in (D_r'')$ and DF exists n.e. and $DF \geq 0$ a.e. then f is continuous and increasing on $[0,1]$. If DF is the qualitative derivative of Marcus [18], the right derivative, the preponderant derivative, or the selective derivatives of O'Malley [20], the above statement about f is true. The proofs are as those in [17], [3] and [20].

See [7] for an additional monotonicity theorem.

CHAPTER IV - MONOTONICITY AND FORAN'S CONDITION (M). APPLICATIONS.

Lemma 6. *Let* $f:[0,1]\longrightarrow R$ *be a continuous function. Let* $P = \{x : f'(x)\geqslant 0\}$. *For any* $a,b\in[0,1]$, *if* $a<b$, $f(a)<f(b)$ *and* $|f(P\cap[a,b])| = 0$ *then for each* $c\in[0,1)$ *there exist perfect non-dense sets* P_c *and* Q_c *such that:* a) $P_c\subset[a,b]$ *and* $Q_c\subset[f(a),f(b)]$;b) $P_{c_1}\cap P_{c_2} = \emptyset$; c) $f/_{P_c}$ *is increasing*; d) $f(P_c) = Q_c$; e) $|Q_c|\geqslant (f(b)-f(a))/2$.

Proof. Let $K_1 = [a_1,b_1]$ and $K_2 = [a_2,b_2]$ be two intervals. If $a_1<b_1<a_2<b_2$ then we denote this by $K_1\prec K_2$. Let $(e_i)_{i=1}^{\infty}$, $e_i\in(0,1)$ be a sequence of real numbers such that $(1-e_1)\cdot(1-e_2)\cdot\ldots\geqslant 1/2$. We shall construct the sets P_c and Q_c by a transfinite process. Suppose that $a<b$, $[a,b]\subset[0,1]$, $f(a)<f(b)$ and $|f(P\cap[a,b])| = 0$.

Step 1. We show that there exists a positive integer m_1 such that if $\mathcal{A}=\{1,2,\ldots,m_1\}$ then the rectangles $D_{i_1}^{c_1} = K_{i_1}^{c_1}\times J_{i_1}$, $i_1\in\mathcal{A}$, $c_1\in\{0,1\}$, have the following properties:

(i) $K_{i_1}^{c_1}$ is a closed subinterval of $[a,b]$; $K_i^1\prec K_j^0$, for $i,j\in\mathcal{A}$, $i<j$, and $K_i^0\prec K^1$, for $i\in\mathcal{A}$.

(ii) J_{i_1} is a closed subinterval of $[f(a),f(b)]$; $J_i\prec J_j$, for $i,j\in\mathcal{A}$, $i<j$;

(iii) $B(f;K_{i_1}^{c_1})\subset D_{i_1}^{c_1}$; The left side lower corner and the right side upper corner of $D_{i_1}^{c_1}$ belong to $B(f;K_{i_1}^{c_1})$; $\sum_{i_1}|J_{i_1}|> (f(b)-f(a))\cdot(1-e_1)$.

Step 2. For $i_1\in\mathcal{A}$, $c_1\in\{0,1\}$ there exists a positive integer $m_2(i_1c_1)$ such that, if $\mathcal{A}(i_1c_1) = \{1,2,\ldots,m_2(i_1c_1)\}$ then the rectangles $D^{c_1c_2}_{i_1i_2} = K^{c_1c_2}_{i_1i_2} \times J^{c_1}_{i_1i_2}$, $i_2 \in \mathcal{A}(i_1c_1)$, $c_2 \in \{0,1\}$, have the following properties:

(i) $K^{c_1c_2}_{i_1i_2}$ is a closed subinterval of $K^{c_1}_{i_1}$; $K^{c_1,1}_{i_1,i} \prec K^{c_1,0}_{i_1,j}$, for $i,j \in \mathcal{A}(i_1c_1)$, $i<j$, and $K^{c_1,0}_{i_1,i} \prec K^{c_1,1}_{i_1,i}$, for $i \in \mathcal{A}(i_1c_1)$;

(ii) $J^{c_1}_{i_1i_2}$ is a closed subinterval of J_{i_1} ; $J^{c_1}_{i_1,i} \prec J^{c_1}_{i_1,j}$, for $i,j \in \mathcal{A}(i_1c_1)$, $i<j$;

(iii) $B(f;K^{c_1c_2}_{i_1i_2}) \subset D^{c_1c_2}_{i_1i_2}$; The left side lower corner and the right side upper corner of $D^{c_1c_2}_{i_1i_2}$ belong to $B(f;K^{c_1c_2}_{i_1i_2})$;

$\sum_{i_2} |J^{c_1}_{i_1i_2}| \geqslant |J_{i_1}| \cdot (1-e_2)$. Hence $\sum_{i_1} \sum_{i_2} |J^{c_1}_{i_1i_2}| \geqslant (1-e_2) \cdot$

$\sum_{i_1} |J_{i_1}| \geqslant (1-e_1)\cdot(1-e_2)\cdot(f(b)-f(a))$;

Step (n+1). (n≥2). Let $i_1 \in \mathcal{A}, \ldots, i_n \in \mathcal{A}(i_1c_1\ldots i_{n-1}c_{n-1})$, $c_i \in \{0,1\}$ for each i. We show that there exists a positive integer $m_{n+1}(i_1c_1\ldots i_nc_n)$ such that if $\mathcal{A}(i_1c_1\ldots i_nc_n) = \{1,2,\ldots, m_{n+1}(i_1c_1\ldots i_nc_n)\}$ then the rectangles

$$D^{c_1\ldots c_{n+1}}_{i_1\ldots i_{n+1}} = K^{c_1\ldots c_{n+1}}_{i_1\ldots i_{n+1}} \times J^{c_1\ldots c_n}_{i_1\ldots i_n i_{n+1}} , \quad i_{n+1} \in \mathcal{A}(i_1c_1\ldots i_nc_n),$$

have the following properties:

(i) $K^{c_1\ldots c_{n+1}}_{i_1\ldots i_{n+1}}$ is a closed subinterval of $K^{c_1\ldots c_n}_{i_1\ldots i_n}$; For $i<j$, $i,j \in \mathcal{A}(i_1c_1\ldots i_nc_n)$ we have $K^{c_1\ldots c_n,1}_{i_1\ldots i_n,i} \prec K^{c_1\ldots c_n,0}_{i_1\ldots i_n,j}$; For $i \in \mathcal{A}(i_1c_1\ldots i_nc_n)$ we have $K^{c_1\ldots c_n,0}_{i_1\ldots i_n,i} \prec K^{c_1\ldots c_n,1}_{i_1\ldots i_n,i}$;

(ii) $J_{i_1\dots i_n i_{n+1}}^{c_1\dots c_n}$ is a closed subinterval of $J_{i_1\dots i_{n-1} i_n}^{c_1\dots c_{n-1}}$; For $i<j$, $i,j\in \mathcal{A}(i_1c_1\dots i_nc_n)$, $J_{i_1\dots i_n,i}^{c_1\dots c_n} \prec J_{i_1\dots i_n,j}^{c_1\dots c_n}$;

(iii) $B(f;K_{i_1\dots i_{n+1}}^{c_1\dots c_{n+1}}) \subset D_{i_1\dots i_{n+1}}^{c_1\dots c_{n+1}}$; The left side lower corner and the right side upper corner of $D_{i_1\dots i_{n+1}}^{c_1\dots c_{n+1}}$ belong to $B(f;K_{i_1\dots i_{n+1}}^{c_1\dots c_{n+1}})$; $\sum_{i_{n+1}} |J_{i_1\dots i_n i_{n+1}}^{c_1\dots c_n}| \geqslant |J_{i_1\dots i_{n-1} i_n}^{c_1\dots c_{n-1}}|\cdot(1-e_{n+1})$, hence $\sum_{i_1}\sum_{i_2}\dots\sum_{i_{n+1}} |J_{i_1\dots i_n i_{n+1}}^{c_1\dots c_n}| \geqslant (1-e_1)\cdot(1-e_2)\cdot\dots\cdot(1-e_{n+1})\cdot$ $\cdot(f(b)-f(a))$.

Now we can define the sets P_c and Q_c. Let $c\in[0,1)$, then there exist $c_i\in\{0,1\}$ such that c is uniquely represented by $\sum_{i=1}^{\infty} c_i/2^i$. (We choose the infinite representation when two different representations exist.) Then

$$P_c = (\bigcup_{i_1} K_{i_1}^{c_1}) \cap (\bigcup_{i_1}\bigcup_{i_2} K_{i_1 i_2}^{c_1c_2}) \cap \dots \cap (\bigcup_{i_1}\dots\bigcup_{i_n} K_{i_1\dots i_n}^{c_1\dots c_n}) \cap \dots \quad \text{and}$$

$$Q_c = (\bigcup_{i_1} J_{i_1}) \cap (\bigcup_{i_1}\bigcup_{i_2} J_{i_1 i_2}^{c_1}) \cap \dots \cap (\bigcup_{i_1}\dots\bigcup_{i_n} J_{i_1\dots i_{n-1} i_n}^{c_1\dots c_{n-1}}) \cap \dots .$$

It follows that P_c and Q_c have the desired properties. It remains to show that the facts stated in step 1, step 2,..., are true. It suffices to show step 1. It is known that f(P) is a measurable set. Let $E_y = \{x : f(x) = y\}$, $x_y = \inf(E_y \cap [a,b])$ and $A = \{x_y : y \in [f(a),f(b)]\}$. Let $\alpha = \inf(\bar{A})$ and $\beta = \sup(\bar{A})$. Let $A_1 = \{x \in A : f'_+(x) = -\infty\}$. Then we have: 1) $f/_{\bar{A}}$ is increasing; 2) $\bar{A}$ is nowhere dense in (α,β); 3) $\bar{A} = A\cup\{b_n\}$, where $I_n = (a_n,b_n)$ are the intervals contiguous to A with respect to (α,β); 4) $a_n \in A$; 5) If $x \in [a_n,b_n]$ then $f(x) \leqslant f(a_n)$; 6) $f(a_n) = f(b_n)$; 7) $\overline{A}$ is a perfect set; 8) If $x \in A$ then $f'_-(x) \geqslant 0$; 9) $|f(A_1)| = f(b)-f(a)$.

The justifications of 1) through 9) are brief:

1) By the definition of A and the continuity of f it follows that $f|_A$ is increasing. Applying again the continuity of f it follows that $f|_{\overline{A}}$ is increasing.

2) Suppose on the contrary that $\overline{A} \supset [\alpha_1,\beta_1]$, $\alpha_1 \neq \beta_1$. Then by 1), $f|[\alpha_1,\beta_1]$ is increasing. Hence $f|[\alpha_1,\beta_1]$ is VB_*. Suppose that $f(\alpha_1) < f(\beta_1)$ then by [21] (Theorem 7.2, p.230), $|f(P \cap [a,b])| \geqslant f(\beta_1)-f(\alpha_1) > 0$. This contradicts the fact that $|f(P \cap [a,b])| = 0$. Therefore $f(\alpha_1) = f(\beta_1) = \gamma$. Since $x_\gamma \leqslant \alpha_1$ it follows that $(\alpha_1,\beta_1) \cap A = \emptyset$. Hence $(\alpha_1, \beta_1) \cap \overline{A} = \emptyset$, which contradicts our supposition. Therefore $\overline{A}$ is nowhere dense in (α,β).

3) Clearly $\overline{A} \supset A \cup \{b_n\}$. Conversely, let $x \in \overline{A}-A$ and let $y = f(x)$. Then $x_y \in A$ and $x_y < x$. Suppose on the contrary that $(x_y,x) \cap \overline{A} \neq \emptyset$, then there exists $x_1 \in (x_y,x) \cap \overline{A}$. By 1), $f|_{\overline{A} \cap [x_y,x]}$ is constant, hence $f(x_y) = f(x_1) = f(x)$. Since $x_1 \in \overline{A}$ there is an $x_2 \in (x_y,x)$ such that $x_2 \in A$ and $f(x_2) = f(x_y)$, hence $x_2 = x_y$, a contradiction. Thus $(x_y,x) \cap \overline{A} = \emptyset$ and (x_y,x) is an interval contiguous to A with respect to (α,β), namely $I_n = (a_n,b_n)$, for some n. Hence $x = b_n$ and $\overline{A} = A \cup \{b_n\}$.

4) Suppose on the contrary that $a_n \notin A$. Then by 3), $a_n = b_k$, for some $k \in \mathbb{N}$. Then $(a_k,b_k) \cap \overline{A} = \{a_n\}$. It follows that a_n is an isolated point of A. Hence $a_n \in A$, a contradiction. Thus $a_n \in A$.

5) Suppose on the contrary that there exists $x_n \in (a_n,b_n]$ such that $f(x_n) > f(a_n)$. Let $\gamma_n = (f(x_n)+f(a_n))/2$. By 1) and 4), since $x_{\gamma_n} \in A$ it follows that $a_n < x_{\gamma_n} < b_n$. Indeed, $a_n \in A$ (by 4)), $x_{\gamma_n} \in A$, $x_{f(x_n)}$

$\in A$. By 1), since $f(a_n) < f(x_{\gamma_n}) < f(x_n)$, it follows that $a_n < x_{\gamma_n} < x_{f(x_n)} \leqslant x_n \leqslant b_n$. Hence $x_{\gamma_n} \in (a_n, b_n) \cap A$. Contradiction.

6) By 5), $f(b_n) \leqslant f(a_n)$ and by 1), $f(a_n) \leqslant f(b_n)$. Thus $f(a_n) = f(b_n)$.

7) Suppose on the contrary that there exists $x_0 \in (\alpha, \beta) \cap \bar{A}$, isolated in A. Then there exist two intervals contiguous to $\bar{A}$, I_j and I_k, such that $x_0 = b_j = a_k$. By 6), $f(a_j) = f(a_k)$ and by 4), $a_j, a_k \in A$, a contradiction.

8) Let $x \in A$. Then $f'_-(x) = \lim\inf (f(x') - f(x))/(x' - x)$, $x' \longrightarrow x$, $x' < x$. Suppose on the contrary that $f(x') > f(x)$, for $x' - x < 0$. Then $x' \geqslant x_{f(x')} > x$ (by 1)) and $x' > x$, a contradiction. Hence $f(x') - f(x) \leqslant 0$.

9) For each $x \in A - A_1$, $\underline{f}(x) > -\infty$ (this follows by 8)). By [21] (Theorem 10.1, p.234), f is VBG_* on $A - A_1$. Let $B = \{x \in A - A_1 : f'(x)$ exists finite or infinite at $x\}$. By 8), $B \subset P \cap [a,b]$. Hence $|f(B)| = |f(P \cap [a,b])| = 0$. Let $B_1 = (A - A_1) - B$. Then by [21] (Theorem 7.2, p.230), $|f(B_1)| = 0$, hence $|f(A - A_1)| = 0$. Since $f(A) = [f(a), f(b)]$ it follows that $|f(A_1)| = f(b) - f(a)$.

Now we cover the set $f(A_1)$ with a collection of closed intervals in the Vitali sense: Let $x \in A_1$, $\varepsilon > 0$ and $\delta(x,\varepsilon) > 0$ be such that $f([x, x+\delta(x,\varepsilon)]) \subset [f(x) - \varepsilon/2, f(x) + \varepsilon/2]$. By 1), f is increasing on $\bar{A} \cap [x, x+\delta(x,\varepsilon)]$. Since $f'_+(x) = -\infty$, it follows that there exists $y \in [x, x+\delta(x,\varepsilon)] - \bar{A}$ such that $f(x) > f(y)$. Let $n(x,y,\varepsilon)$ be a positive integer such that $y \in I_{n(x,y,\varepsilon)}$. Let $m_{n(x,y,\varepsilon)} = \inf\{f(t) : t \in I_{n(x,y,\varepsilon)}\}$. Let $c^1(x,y,\varepsilon) = \inf\{t \in I_{n(x,y,\varepsilon)} : f(t) = m_{n(x,y,\varepsilon)}\}$ and $d^1(x,y,\varepsilon) = b_{n(x,y,\varepsilon)}$; $c^0(x,y,\varepsilon) = A \cap E_{m_{n(x,y,\varepsilon)}}$; $d^0(x,y,\varepsilon) = a_{n(x,y,\varepsilon)}$; $J(x,y,\varepsilon) = [f(c^0(x,y,\varepsilon)), f(d^0(x,y,\varepsilon))] = [f(c^1(x,y,\varepsilon)), f(d^1(x,y,\varepsilon))]$. Then $f(x) \in J(x,y,\varepsilon)$ and $|J(x,y,\varepsilon)| < \varepsilon$.

Let $K^0(x,y,\varepsilon) = [c^0(x,y,\varepsilon), d^0(x,y,\varepsilon)]$; $K^1(x,y,\varepsilon) = [c^1(x,y,\varepsilon), d^1(x,y,\varepsilon)]$. Then $f(A_1) \subset \bigcup_{x \in A_1} J(x,y,\varepsilon)$. By [21] (Vitali's theorem, p.109), there exist a natural number m_1 and intervals $J_1 \prec \dots \prec J_{m_1}$ such that $J_1, \dots, J_{m_1} \in \bigcup_x J(x,y,\varepsilon)$ (therefore we have (ii)) and $\sum_{i_1=1}^{m_1} |J_{i_1}| > (f(b)-f(a)) \cdot (1-e_1)$ (we have (iii)). Now we have the corresponding intervals $\{K^0_{i_1}\}$, $i_1 = 1,2,\dots,m_1$, and $\{K^1_{i_1}\}$, $i_1 = 1,2,\dots,m_1$. By 1), we have (i).

Theorem 9. Suppose that $f:[0,1] \longrightarrow R$ is a continuous function which satisfies $[\overline{M}]$ on $[0,1]$. Then f is derivable on a set of positive measure. Moreover, if there exist $0 \leq a < b \leq 1$ such that $f(a) < f(b)$ then $|f(P)| > 0$, where $P = \{x : f'(x) \geq 0\}$.

Proof. If f is decreasing the proof is obvious. Suppose that f is not decreasing on $[0,1]$. Then there exist $a, b \in [0,1]$, $a < b$ such that $f(a) < f(b)$. Suppose on the contrary that $|f(P \cap [a,b])| = 0$. Then by Lemma 6, there exist infinitely many sets P_t and Q_t such that $|P_t| = 0$, $f/_{P_t}$ is increasing, $f(P_t) = Q_t$ and $|Q_t| > (f(b)-f(a))/2$. By Theorem 3, $f/_{P_t}$ is AC. Hence $|Q_t| = 0$, a contradiction. Therefore, if $f(a) < f(b)$ then $|f(P)| > 0$. By Remark 1,e) and Theorem 6 it follows that $|f(E^{+\infty})| = 0$, where $E^{+\infty} = \{x : f'(x) = +\infty\}$. By [21] (p.236), $|E^{+\infty}| = 0$, hence $|P - E^{+\infty}| = |P| > 0$ (since $f/_{P-E^{+\infty}} \in (N)$), ([21], Theorem 4.6, p.271).

Theorem 10. If a continuous function $f:[0,1] \longrightarrow R$ satisfies $[\overline{M}]$ on $[0,1]$ and if $f'(x) \leq 0$ at almost every point x where $f'(x)$ exists and is finite then f is decreasing on $[0,1]$.

Proof. Suppose that $f \in [\overline{M}]$ and there exist $a,b \in I$, $a<b$ such that $f(a)<f(b)$. Let $P = \{x : +\infty \geq f'(x) \geq 0\}$; $P_0 = \{x : f'(x) = 0\}$; $E_+ = \{x : 0< f'(x) <+\infty\}$; $E_{+\infty} = \{x : f'(x) = +\infty\}$. Then $P = P_0 \cup E_+ \cup E_{+\infty}$, $|f(P_0)| = 0$ (see [21], Theorem 4.5, p.271), $|E_+| = |E_{+\infty}| = 0$ (by hypothesis). By [21] (Theorem 4.6, p.271), $f \in (N)$ on E_+, hence $|f(E_+)| = 0$. By Remark 1,e) and Theorem 6 it follows that $|f(E_{+\infty})| = 0$. Thus $|f(P)| = 0$ which contradicts Theorem 9.

Corollary 3. (An extension of a theorem of Nina Bary - [1], p. 199 or [21], p.286). If a continuous function f satisfies Foran's condition (M) on $[0,1]$ and if $f'(x) \geq 0$ at almost every point x where $f'(x)$ exists and is finite, then f is AC and increasing on $[0,1]$.

Theorem 11. Let $f:[0,1] \longrightarrow R$ be a function belonging to $uCM \cap B_1^* \cap [\underline{M}]$. If $f'(x) \geq 0$ a.e. where $f'(x)$ exists and is finite then f is increasing on $[0,1]$.

To prove this theorem we need the following lemma.

Lemma 7. Let Q be a nonempty perfect set. Let $a = \inf(Q)$, $b = \sup(Q)$. Let $I_n = (a_n,b_n)$ be the intervals contiguous to Q with respect to $[a,b]$. Let f be a function defined on $[a,b]$, with the following properties: (i) $f \in \mathscr{C}$ on Q; (ii) $f(a_n) \leq f(b_n)$; (iii) $f(I_n) \subset [f(a_n),f(b_n)]$. Let f_1 be a continuous function on $[a,b]$, defined as follows: $f_1(x) = f(x)$, $x \in Q$; $f_1(x) = (x-a_n)\cdot(f(b_n) - f(a_n))/(b_n-a_n) + f(a_n)$, $x \in (a_n,b_n)$. Let $E = \{x \in Q : f'(x)$ exists finite or infinite$\}$; $E_1 = \{x \in Q : f_1'(x)$ exists finite or infinite$\}$; $T = (E-E_1) \cup (E_1 - E)$. Then we have: a) If $A \subset Q$ then $f|_A \in VBG_*$ if and only if $f_1|_A \in VBG_*$; b) $|f(T)| = |T| = 0$ and $f'(x)=f_1'(x)$ a.e. on E.

Proof. Let $c,d \in Q$. By (ii) and (iii) we have:

(4) $\quad O(f; [c,d]) = O(f_1; [c,d])$.

Let $A \subset Q$. Then $f|_A \in VB_*$ if and only if $f_1|_A \in VB_*$ (by (4) and by the definition of VB_*). Hence $f|_A \in VBG_*$ if and only if $f_1|_A \in VBG_*$, and we have a). Let $T_1 = E_1 - E$, $T_2 = E - E_1$. We show that $|f(T_1)| = |T_1| = 0$. We have $f_1|_{T_1} \in VBG_*$ (see [21], Theorem 10.1,p.234). By a), $f|_{T_1} \in VBG_*$. Since $f'(x)$ does not exist (finite or infinite) on T_1, by [21] (p.230, Theorem 7.2), $|f(T_1)| = \Lambda(B(f;T_1)) = 0$. Similarly, $|f(T_2)| = \Lambda(B(f;T_2))$. ($\Lambda(X)$ is the Hausdorff length of X.) Clearly $|E| = |E_1| = |E \cap E_1|$. Since $f = f_1$ on Q and Q is perfect, it follows that $f'(x) = f_1'(x)$ on $E \cap E_1$.

Proof of Theorem 11. Suppose that $f \in uCM \cap B_1^* \cap [\underline{M}]$. Since $f \in B_1^*$ on $[0,1]$, there exists a sequence of intervals I_n whose union is dense in $[0,1]$ and on each of which f is continuous. By Theorem 10, f is increasing on I_n. Hence $f|_{\overline{I}_n}$ is increasing. The intervals I_n can be chosen to be maximal open intervals of monotonicity of f. We wish to show that in fact there exists only one such maximal open interval, namely the interior of $[0,1]$. Suppose that there is more than one such maximal interval and let $Q = [0,1] - (\bigcup I_n)$. The set Q is a perfect subset of $[0,1]$, for Q is obviously closed and if x_0 is isolated in Q then f would be increasing on some $\overline{I}_j$ (since $f \in uCM$), having x_0 as a right-hand endpoint, and some interval $\overline{I}_k$ (since $f \in uCM$), having x_0 as a left-hand endpoint. Then f is increasing on $I_j \cup I_k \cup \{x_0\}$, that would contradict the maximality of the intervals I_j and I_k. By Baire's Category theorem, there exist $a,b \in [0,1]$, $a<b$, such that $Q \cap (a,b) \neq \emptyset$ and $f|_{Q \cap [a,b]}$ is continuous. Let $f_1(x) = f(x)$, $x \in Q \cap [a,b]$. Extending f_1 linearly on the closure of each interval contiguous to Q we have f_1 defined and continuous on $[a,b]$. Also $f_1 \in [\underline{M}]$ on $[a,b]$. Indeed, let $A = \overline{A} \subset [a,b]$ be such that $f_1|_A \in VB$. Then $f|_{A \cap Q} \in VB \cap \mathcal{C}$. Since $f \in [\underline{M}]$

it follows that $f \in \underline{AC}$ on $A \cap Q$. Hence $f_1 \in [\underline{ACG}]$ on A. By Lemma 1, $f_1 \in \underline{AC}$ on A. Hence $f_1 \in [\underline{M}]$. By Lemma 7, $f_1'(x) = f'(x)$ a.e. on E. Since $f'(x) \geq 0$ a.e. on E, it follows that $f_1'(x) \geq 0$ a.e. where $f_1'(x)$ exists on Q. On each interval contiguous to Q, f_1 is increasing. Hence $f_1'(x) \geq 0$ a.e. where $f_1'(x)$ exists on $[a,b]$. By Theorem 10, f_1 is increasing on $[a,b]$. Hence $f|_{Q \cap [a,b]}$ is increasing. But $f|_{\overline{I}_n}$ is also increasing (since $f \in uCM$). Thus f is increasing on $[a,b]$, a contradiction.

<u>Theorem 12. Let</u> $f:[0,1] \longrightarrow R$ <u>be a</u> DB_1^* <u>function. Let</u> $U(f) = \operatorname{int}\{x : f$ <u>is continuous at</u> $x\}$. <u>Suppose that</u> $f \in [\underline{M}]$ <u>on</u> $U(f)$. <u>If</u> $f'(x) \geq 0$ a.e. <u>on</u> $U(f)$ <u>where</u> f <u>is derivable, then</u> f <u>is continuous and increasing on</u> $[0,1]$.

<u>Proof.</u> Suppose that $f \in [\underline{M}]$. It should be noted that $U(f)$ is a dense open subset of $[0,1]$, since $f \in B_1^*$. First we show that f is increasing on every component of U(f). Let J be a component of U(f) and $[c,d] \subset J$. By Theorem 10, f is continuous and increasing on the interval $[c,d]$. Since $[c,d]$ was an arbitrary subinterval of J, f is increasing on J. Since $f \in D$, $f|_{\overline{J}}$ is continuous and increasing. Suppose that $U(f) \neq (0,1)$. Then $Q = [0,1] - U(f)$ is a perfect set (if necessary without 0 and 1). Since $f \in B_1^*$, there exist $a,b \in Q$, $a<b$, such that $(a,b) \cap Q \neq \emptyset$ and $f|_{Q \cap [a,b]}$ is continuous. It follows that $f|_{[a,b]} \in \mathcal{C}$, hence $(a,b) \subset U(f)$, a contradiction. Therefore $U(f) = (0,1)$ and f is continuous and increasing on $[0,1]$.

<u>Remark 7.</u> If $f'(x)$ is replaced by $f'_{ap}(x)$, Theorem 12 remains true, and this is in fact an extension of Theorem 2 of [6].

<u>Theorem 13. Let</u> $F:[0,1] \longrightarrow R$ <u>be a</u> DB_1^* <u>function and</u> $U(f) = \operatorname{int}\{x : F$ <u>is continuous at</u> $x\}$. <u>Suppose that</u> $F \in [\overline{M}]$ (<u>resp.</u> $[M]$) <u>on</u> $U(F)$. <u>Let</u> $P = \{x : F$ <u>is derivable at</u> x <u>and</u> $F'(x) > 0\} \cap U(F)$. <u>Then</u>

F is $\overline{AC}\cap\mathcal{B}$ (resp. AC) on $[0,1]$ iff F' is summable on P.

Proof. The necessity is obvious. We prove the sufficiency. Let $g(x) = F'(x)$, $x\in P$ and $g(x) = 0$, $x\in [0,1] - P$. Let $G(x) = \int_0^x g(t)dt$. Then $G(x)$ is AC and nondecreasing on $[0,1]$. Let $H(x) = G(x)-F(x)$. Then $H\in DB_1^*$ on $[0,1]$. Let $U(H) = \text{int}\{x : H \text{ is continuous at } x\}$. Then $U(H) = U(F)$. We show that H is increasing and continuous on $[0,1]$. Clearly $H\in[\underline{M}]$ on U(F). Let $x\in U(F)$ be any point at which both F and G are derivable, then H is derivable at x and $H'(x) = G'(x) - F'(x)$. If $x\in P$ then $H'(x) = 0$ and if $x\notin P$ then $F'(x)\leqslant 0$ and $G'(x)\geqslant 0$. Hence $H'(x)\geqslant 0$. Consequently, $H'(x)$ is nonnegative at almost all points x where $H'(x)$ exists on U(F). By Theorem 12, H is increasing and continuous on $[0,1]$. It follows that $F\in VB\cap\mathcal{B}$ on $[0,1]$. By the definition of $[\overline{M}]$ (resp. $[M]$) it follows that $F\in \overline{AC}$ (resp. AC) on $[0,1]$.

Remark 8. Theorem 13 generalizes Theorem 7.7 of [21](p.285) and Theorem 1 of [15](p.261).

Theorem 14. Let $F:[0,1]\longrightarrow R$ be a DB_1^* function and let $U(F) = \text{int}\{x : F \text{ is continuous at } x\}$. Suppose that $F\in[\overline{M}]$ (resp. $[M]$) on U(F). Let $F^*(x) = F'(x)$ if it exists and is finite; otherwise, let $F^*(x) = 0$. Let $F^*_{ap}(x) = F'_{ap}(x)$ if it exists and is finite; otherwise let $F^*_{ap}(x) = 0$. If there exists a continuous function $G:[0,1]\longrightarrow R$ such that:

a) $G\in VBG_*\cap \underline{ACG}$, $G'(x)\geqslant F^*(x)$ a.e. on $[0,1]$, then $F\in VBG_*\cap\overline{ACG}\cap\mathcal{B}$ (resp. $ACG_*\cap\mathcal{B}$) on $[0,1]$;

b) $G\in \Delta_{a.e.}\cap\underline{ACG}$, $G'(x)\geqslant F^*(x)$ a.e. on $[0,1]$, then $F\in\Delta_{a.e.}\cap\overline{ACG}\cap\mathcal{B}$ (resp. $\Delta_{a.e.}\cap ACG\cap\mathcal{B}$) on $[0,1]$;

c) $G\in\underline{ACG}$, $G'_{ap}(x)\geqslant F^*_{ap}(x)$ a.e. on $[0,1]$, then $F\in\overline{ACG}\cap\mathcal{B}$ (resp. $ACG\cap\mathcal{B}$) on $[0,1]$. ($\Delta_{a.e.} = \{F:[0,1]\longrightarrow R,\ F \text{ is derivable a.e.}\}$.

Proof. Let $H(x) = G(x)-F(x)$. Then $H \in [\underline{M}]$ on $U(F) = U(H) = \mathrm{int}\{x : H \text{ is continuous at } x\}$. For a),b), $H'(x) \geq 0$ a.e. on $U(H)$ where $H'(x)$ exists and is finite, and for c), $H'_{ap}(x) \geq 0$ a.e. on $U(H)$ where $H'_{ap}(x)$ exists and is finite. By Theorem 12 and Remark 7, H is increasing and continuous on $[0,1]$. Now $F = G-H \in VBG$ and by the definition of $[\overline{M}]$(resp.$[M]$) it follows that $F \in \overline{ACG}$ (resp.ACG). Clearly for a) and b), $F \in VBG_*$ and $F \in \Delta_{a.e.}$ respectively. Since $VBG_* \cap ACG \cap \mathcal{C} = ACG_*$ on $[0,1]$ (see Theorem 8.8,p.233 of [21]) the proof is complete.

Remark 9. i) Theorem 14 remains true if: 1) "$G'(x) \geq F^*(x)$ a.e. on $[0,1]$" is replaced by "$G'(x) \geq F'(x)$ a.e. on $U(F)$ where $F'(x)$ exists and is finite" in cases a) and b); 2)"$G'_{ap}(x) \geq F^*_{ap}(x)$ a.e. on $[0,1]$" is replaced by "$G'_{ap}(x) \geq F'_{ap}(x)$ a.e. on $U(F)$ where $F'_{ap}(x)$ exists and is finite" in case c).

ii) The second part of Theorem 14 is an extension of a theorem of Saks (see [21],p.286).

iii) The second part of Theorem 14,c) is an extension of Theorem 2 of [12](p.446).

iv) Since an approximately differentiable function F is $DB_1^* \cap (N) \subset DB_1^* \cap [M]$(see [15],p.261), by the second part of Theorem 14,a), we have the following theorem of [14](p.295):

Let $F:[0,1] \longrightarrow R$ be approximately differentiable. If F^* is Perron integrable on $[0,1]$ then F is ACG_* on $[0,1]$.

v) In Theorem 14,b) we cannot give up the condition $\Delta_{a.e.}$ on $[0,1]$ (see Example 2 of [13],p.305).

CHAPTER - MONOTONICITY AND PROPERTIES $[M_*]$, $[\underline{M}_*]$, $[\overline{M}_*]$. APPLICATIONS.

Recall that by Theorem 6 it follows that for Darboux functions on $[0,1]$, $[M_*] = N^{\infty}$, $[\underline{M}_*] = N^{-\infty}$ and $[\overline{M}_*] = N^{+\infty}$.

Theorem 15. If $F:[0,1]\longrightarrow R$ belongs to $D\cap(+)\cap N^{+\infty}$ and $F'(x)\leq 0$ a.e. on $[0,1]$ where $F'(x)$ exists and is finite then F is continuous and decreasing on $[0,1]$.

Proof. By Theorem 2 and by Lemma B, F is decreasing on $[0,1]$. Since $F\in D$ it follows that F is continuous on $[0,1]$.

Corollary 4. Let $F:[0,1]\longrightarrow R$ be a function with the following properties on $[0,1]$: (i) F is measurable and (D_r'') (particularly $F\in DB_1$); (ii) $F\in (N)$; $F'(x)\geq 0$ a.e. where F is derivable. Then F is increasing and AC on $[0,1]$.

Proof. It follows by Remark 1,c),e),k) and Theorem 1,c),e), and by Theorem 15.

Open problem. Note that the second part of Corollary 4 is in fact C.M. Lee's Theorem 1 of [15]. Does C.M. Lee's theorem remain true if condition (N) is replaced by condition [M] ?

Corollary 5. Let $F:[0,1]\longrightarrow R$ be a $DB_1\cap\mathcal{C}\cap\mathfrak{D}$ function. Let $F^*_{\mathfrak{D}}(x) = DF(x)$ if it exists and is finite; otherwise, let $F^*_{\mathfrak{D}}(x)=0$. If $F^*_{\mathfrak{D}}$ is $\mathcal{F}\mathfrak{D}$-integrable on $[0,1]$ then $F\in\mathcal{F}\cap\mathcal{C}\cap\mathfrak{D}$ on $[0,1]$.

Proof. Let $G(x) = \mathcal{F}\mathfrak{D}\int_0^x F^*_{\mathfrak{D}}(t)dt$. Then $DG(x) = DF^*_{\mathfrak{D}}(x)$ a.e. on $[0,1]$ and $G\in\mathcal{F}\cap\mathcal{C}\cap\mathfrak{D}$. Let $H(x) = G(x)-F(x)$. Then $H\in DB_1\cap\mathcal{C}\cap\mathfrak{D}$ and $H'(x) = 0$ a.e. where H is derivable on $[0,1]$. By Corollary 4 (the second part), H is constant on $[0,1]$. Therefore $F\in\mathcal{F}\cap\mathcal{C}\cap\mathfrak{D}$ on $[0,1]$.

Example. There exists a continuous function $g:[0,1]\longrightarrow[0,2]$ with the following properties: (i) $g'(x)$ exists on $[0,1]-C$ (C = the Cantor ternary set); $g'(x)\leqslant 0$ on $[0,1]-C$; if $x\in C$ then $g'(x)$ does not exist (finite or infinite); (ii) $g\in N^{\infty} = N^{+\infty}\cap N^{-\infty}$; (iii) $g\notin (+)\cap(M)$.

Proof. For each $x\in C$, let $g(x) = g(\sum c_i(x)/3^i) = \sum c_{2i}(x)/2^i$. Then g is continuous on C. Extending g linearly on each interval contiguous to C we have g defined and continuous on $[0,1]$. (i) We observe that if I is an interval contiguous to C from the step $2k$ in the Cantor ternary process then g is constant on I and if I is an interval contiguous to C from the step $2k+1$ in the Cantor ternary process then g is strictly decreasing on I. It follows that g is derivable on $[0,1]-C$ and $g'(x)\leqslant 0$ on $[0,1]-C$. Let $x_o\in C$ and let $c_i\in\{0,2\}$, such that $x_o = \sum c_i/3^i$. Let $\{x_n\}$ and $\{y_n\}$, $x_n, y_n \in C$, be two sequences which converge to x_o: $x_n = \sum_{i\neq 2n+1} c_i/3^i + (2-c_{2n+1})/3^{2n+1}$, $y_n = \sum_{i\neq 2n} c_i/3^i + (2-c_{2n})/3^{2n}$. Clearly $|x_n-x_o| = 2/3^{2n+1}$, $|y_n-x_o| = 2/9^n$, $g(x_n) = g(x_o)$ and $|g(y_n)-g(x_o)| = 2/2^n$. Hence $|(g(x_n)-g(x_o))/(x_n-x_o)| = 0$ and $\lim|(g(y_n)-g(x_o))/(y_n-x_o)| = +\infty$, $n\longrightarrow+\infty$. Therefore $g'(x_o)$ does not exist.

(ii) by (i), $E^{\infty} = \emptyset$. Hence $g\in N^{\infty}$.

(iii) Let Q be a perfect subset of C, $Q = \{x\in C : c_{2n+1}(x) = 0, n= 1,2,\ldots\}$. Clearly $g(Q) = [0,2]$. We show that g is increasing on Q. Let $x,y\in Q$, $x<y$ and let m be the first natural number such that $c_{2m}(x)<c_{2m}(y)$. Then $c_i(x) = c_i(y)$, $i = 1,2,\ldots,2m-1$ and $g(y)-g(x) \geqslant 2/2^m - \sum_{i=1}^{\infty} 2/2^{m+i} = 0$. Hence $g(y)\geqslant g(x)$. By Theorem A, $g\notin(M)$.

Since $[g(0),g(1/3)] = [0,2]$ and $|g(P\cap[0, 1/3])| = 0$, it follows that $g\in(+)$. ($P = \{x : g'(x)\geq 0\}$.)

Remark 10. The Example shows that we cannot give up the condition (+) in Theorem 15.

Theorem 16. Let $h:[0,1]\longrightarrow R$ be a function belonging to $(DB_1\cap T_2\cap N^{+\infty})\boxplus(\mathcal{C}\cap VBG_*\cap N^{+\infty})$. If $h'(x)\leq 0$ a.e. where h is derivable, then h is continuous and decreasing on $[0,1]$.

Proof. Let $f,g:[0,1]\longrightarrow R$, $f\in(DB_1\cap T_2\cap N^{+\infty})$ and $g\in(\mathcal{C}\cap VBG_*\cap N^{+\infty}) = (\mathcal{C}\cap VBG_*\cap\overline{ACG})$, such that $h = f+g$ on $[0,1]$. By Remark 1, i), $h\in DB_1$ on $[0,1]$. For g there exists a sequence of intervals I_n whose union is dense in $[0,1]$ and on each of which $g\in\overline{AC}\cap\mathcal{C}$. Let $[c_n,d_n]\subset I_n$. Since $h\in(DB_1\cap T_2\cap N^{+\infty})\boxplus(\overline{AC}\cap\mathcal{C})$ on I_n, by Corollary 2,b), $f\in\overline{AC}$ on $[c_n,d_n]$, hence $h\in\overline{AC}$ on $[c_n,d_n]$. By Lemma B, h is decreasing on $[c_n,d_n]$. Since $h\in D$, h is continuous and decreasing on $\overline{I}_n$. The intervals I_n can be chosen to be maximal open intervals of monotonicity of h. Suppose that $Q = [0,1]-(\bigcup I_n)\neq\emptyset$. Then Q is a perfect nonempty subset of $[0,1]$ (if necessary without 0 and 1). Let $0\leq a<b\leq 1$ such that $(a,b)\cap Q = \emptyset$ and $g|_{[a,b]\cap Q}\in\overline{AC}\cap VB_*$. Let $g_1(x) = g(x)$, $x\in[a,b]\cap Q$ and let g_1 be linear on the closure of each interval contiguous to Q with respect to $[a,b]$. Then $g_1\in VB_*\cap\mathcal{C}$ on $[a,b]$. Let $f_1(x) = h(x) - g_1(x)$. By Remark 1,i), $f_1\in DB_1$ on $[a,b]$. Since $f\in T_2$ and $f_1\in\overline{AC}$ on each I_n, it follows that $f_1\in T_2$ on $[a,b]$ and $f_1\in N^{+\infty}$ on each I_n. Let $Q_1 = \overline{Q}_1\subset[a,b]\cap Q$ such that $f_1|_{Q_1}\in VB_*$. Since $g_1\in VB_*$ on $[a,b]$ it follows that $h = f_1+g_1\in VB_*$ on Q_1. Hence $f = h-g\in VB_*$ on Q_1. Since $f\in N^{+\infty}$, by Lemma A, $f|_{Q_1}$ is $\overline{AC}$. Hence $f_1|_{Q_1}\in\overline{AC}$. It follows that $f_1\in[\overline{M}_*]$ on $Q\cap[a,b]$. Hence $h = f_1+g_1\in(DB_1\cap T_2\cap N^{+\infty})\boxplus(\overline{AC}\cap\mathcal{C})$ on $[a,b]$. Now h is decreasing on $[a,b]$, a contradiction.

Corollary 6. Let $F,G:[0,1]\longrightarrow R$ be two functions such that $F\in DB_1\cap T_2\cap N^{+\infty}$ (resp. $DB_1\cap T_2\cap N^{\infty}$), $G\in ACG\cap \mathcal{B}\cap VBG_*$ and $G'(x)\geq F'(x)$ a.e. where F is derivable. Then $F\in \overline{ACG}\cap VBG_*\cap \mathcal{B}$ (resp. $ACG_*\cap \mathcal{B}$) on $[0,1]$ and $H = F-G$ is continuous and decreasing on $[0,1]$.

Proof. Clearly $H'(x)\leq 0$ a.e. on $[0,1]$ where H is derivable. By Theorem 16, H is decreasing and continuous on $[0,1]$. Hence $F\in VBG_*\cap \mathcal{B}$. Now $F\in \overline{ACG}$ (resp. ACG_*) on $[0,1]$.

Remark 11. By Corollary 6 we have the following theorem:

Let $F:[0,1]\longrightarrow R$, $F\in DB_1\cap T_2\cap N^{\infty}$. If F^* (see Theorem 14) has a major function in the Perron sense then $F\in ACG_*\cap \mathcal{B}$ on $[0,1]$.

This theorem is an extension of a theorem of Saks (see [21], p.286). (See also Remark 9,ii).)

Corollary 7. Let $h:[0,1]\longrightarrow R$ be a function belonging to $(DB_1\cap(N))\boxplus(ACG_*\cap \mathcal{B})$ on $[0,1]$. If $h'(x)\geq 0$ a.e. where h is derivable then $h\in AC$ and is increasing on $[0,1]$.

Remark 12. In [20], Mazurkiewicz has constructed a continuous function $f(x)$ on $[0,1]$ such that for $b\neq 0$ the function $f(x)+bx$ does not satisfy Lusin's condition (N). Therefore $DB_1\cap(N)\subset(DB_1\cap(N))\oplus(ACG_*\cap \mathcal{B})\subset DB_1\cap[M]$ on $[0,1]$. Thus Corollary 7 is a partial answer for the Open problem.

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