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NONSTANDARD TECHNIQUES IN REAL ANALYSIS

1. Introduction.

The purpose of this paper is to display the techniques of non-standard analysis in simplicity sufficient to make them readily available to those working on problems in differentiation and related topics. While the foundations of non-standard analysis lie within the field of mathematical logic, several relatively recent books and papers ([1], [3], [4]) reduce the complexity of the logic considerably removing the mystery from the analytic content.

The first seven sections concern only analysis in $\mathbb{R}$, the set of real numbers. Sections 3, 4 and 7 introduce, each, one of the three major elementary operating tools: the transfer principle, internality, and concurrence. Section 9 introduces a fourth tool - saturation. Most of the examples in Sections 1 through 5 are well-known non-standard examples to illustrate the technique being studied.

The applications in Sections 6 through 9 are original. In Sections 6 and 8, a non-standard and more broadly applicable version of Leinfelder's "locally valid suborderings" is given (cf. [2]). In Section 9, Wattenberg's non-standard characterization of approximately continuous functions (cf. [6]) is extended to other types of continuity using the concept of "system of paths" introduced by O'Malley, Thomson and Bruckner (cf. [5]).

2. Hyperreal Numbers.

Initially, the set $\mathbb{R}$ of real numbers is enlarged to a set $^*\mathbb{R}$ called the hyperreal numbers. Intuitively, a new set of objects called infinitesimals is attached to the real number zero forming an "infinitesimal neighborhood" called a monad (or halo). This is translated by addition to create a monad for each $x \in \mathbb{R}$. In addition, infinitely large "numbers" are added. To define such a set $^*\mathbb{R}$ is no great difficulty. To do it in such a way as to preserve the operational calculus of $\mathbb{R}$ in a "meaningful" way is the difficult part. We begin the development.

Since we want to develop the nonstandard ideas in sufficient generality to include not only $\mathbb{R}$, but topological spaces and other structure, we replace $\mathbb{R}$ by some unspecified infinite set S.

Let Γ be an infinite index set and $\mathcal{F}$ an ultrafilter on Γ. (Their specification produces various properties which we discuss later.) Let μ be the measure on Γ induced by the ultrafilter, i.e., $\mu(A) = 1$ if $A \in \mathcal{F}$, $\mu(A) = 0$ if $A \notin \mathcal{F}$.

Let Z_0 be the set of functions mapping Γ into S and denote, for $\delta \in \Gamma$, $f(\delta)$ by f_δ. Write $f \sim g$ in case $f_\delta = g_\delta$ μ - a.e. Since $\mathcal{F}$ is an ultrafilter, " $\sim$ " is an equivalence relation on Z_0. Denote the equivalence class to which f belongs by $\bar{f}$. Define $^*S = \{\bar{f} : f \in Z_0\}$.

With $S = \mathbb{R}$, and Γ and $\mathcal{F}$ properly specified (details of no value to us here), $^*\mathbb{R}$ is called the hyperreal numbers. The constant map gives a natural embedding of $\mathbb{R}$ in $^*\mathbb{R}$.

Proof of the following can be found in [3] or [1]. We use it to develop some of the elementary properties of $^*\mathbb{R}$.

<u>Theorem 2.1</u>. $^*\mathbb{R}$ is a commutative, non-Archimedean ordered field containing $\mathbb{R}$ as an ordered subfield.

Let $\mathbb{F} = \{x \in {^*\mathbb{R}} : |x| \leq y$ for some $y \in \mathbb{R}\}$. $\mathbb{F}$ is called the <u>finite</u> elements of $^*\mathbb{R}$. It is easily shown that $\mathbb{R} \subset \mathbb{F} \subset {^*\mathbb{R}}$, each being a subring of the larger. Since $^*\mathbb{R}$ is non-Archimedean, $^*\mathbb{R} - \mathbb{F}$ is non-empty. Its members are the infinite elements of $^*\mathbb{R}$.

Let $I = \{x \in {^*\mathbb{R}} : 1/x$ is infinite$\}$. I is a non-empty ideal in $^*\mathbb{R}$. Moreover

<u>Theorem 2.2</u>. $\mathbb{F} = \mathbb{R} \oplus I$.

<u>Proof</u>: $\mathbb{R} \cap I = \emptyset$ is clear. Given $a \in \mathbb{F}$, let $A = \{r \in \mathbb{R} : r \leq a\}$. It is easily shown that $b = \text{lub } A$ satisfies $a-b \in I$.

While the above is a very elementary result about $^*\mathbb{R}$ it has simple consequences useful in describing the monads of real numbers. Proofs are easy and left to the reader.

<u>Theorem 2.3</u>. Let $x \sim y$ denote "$x-y \in I$". Then

(1) For each $x \in \mathbb{F}$, there is $y \in \mathbb{R}$ such that $x \sim y$ (y is called the standard part of x. We write $y = st(x)$).

(2) " $\sim$ " is transitive: $x \sim y$, $y \sim z \Rightarrow x \sim z$.

(3) If $x.y$ are in ${}^*\mathbb{R}$ and $|x-y| < \varepsilon$ for $\varepsilon > 0$, ε in ${}^*\mathbb{R}$, then $x \sim y$.

3. Operating in ${}^*\mathbb{R}$: The Transfer Principle.

What has been done so far defines the set in which analysis with infinitesimals is carried out but gives us no set of rules for determining the legitimacy of operations. The value of Robinson's theory of "enlargements" is that a universe is attached to $\mathbb{R}$ in stages and, along with it, a language. The universe includes the sets, functions, relations and operations through which ordinary analysis is carried out: the subject matter, so to speak. The language is a carefully organized semantic structure which tells precisely what can be said that is true.

The key to operating in the $*$-space develops at this point. A parallel universe and language are created for ${}^*\mathbb{R}$, isomorphic at each stage to those developed for $\mathbb{R}$. In this way, the analysis for $\mathbb{R}$ is "transferred" to ${}^*\mathbb{R}$ with a well-defined test for determining the truth of statements about ${}^*\mathbb{R}$.

Details of this development are given in Section 4. What we do here is list some operating rules which display the manner in which the transfer of "truth" occurs, and then illustrate this in the style of [3].

Rules (1). Every subset $A \subset \mathbb{R}$ has a natural extension ${}^*A \subset {}^*\mathbb{R}$ such that $A \subset {}^*A$.

(2) Every function $f : A \to \mathbb{R}$ has a natural extension ${}^*f : {}^*A \to {}^*\mathbb{R}$. (More generally, every n-ary relation on $\mathbb{R}$ has such an extension).

(3) All standard algebraic operations in $\mathbb{R}$ transfer to ${}^*\mathbb{R}$. (No new notation is adopted.)

To understand the transfer principle, it is useful to distinguish between the constant terms, variable terms and logical connectors in mathematical statements.

Consider for example, the statement "Every non-empty subset of $\mathbb{N}$ has a smallest element". It is more formally written

$$\vdash \ (\forall A \in P(\mathbb{N}))(A \neq \phi)(\exists n_0 \in \mathbb{N})(\forall n \in A)(n_0 \leq n)$$

where " $\models$ " indicates it is a valid statement in the language of $\mathbb{R}$.

$\mathbb{P}(\mathbb{N})$, $\mathbb{N}$ and ϕ are constants; n_0,n and A are variables; the rest of the symbols are logical expressions.

For any statement α in the language of $\mathbb{R}$, ${}^*\alpha$ is the statement in the language of ${}^*\mathbb{R}$ obtained by replacing all constants with their natural extensions (i.e., by "starring" them). The "transferred" statement becomes

$$^*\models (\forall A \in {}^*\mathbb{P}(\mathbb{N}))(A \neq {}^*\phi)(\exists n_0 \in {}^*\mathbb{N})(\forall n \in A)(n_0 \leq n).$$

Note that this does not say "every non-empty subset of ${}^*\mathbb{N}$ has a smallest element" (in fact, a false statement). But, if you replace the word "subset" in this statement by "subset of the form ${}^*B$ for some $B \subset \mathbb{N}$", a valid statement in ${}^*\mathbb{R}$ is created.

With this understanding of the symbol ${}^*\alpha$ we are able to state the

Transfer Principle. Let α be a sentence in the language $\mathcal{L}$ of $\mathbb{R}$. Then

$$^*\models {}^*\alpha \quad \text{if and only if} \quad \models \alpha.$$

For a structure S more generally it would read

$$^*\models {}^*\alpha \quad \text{in } {}^*S \text{ if and only if } \models \alpha \quad \text{in } S.$$

Even after giving details in Section 4, we omit the proof of this relatively deep theorem. See [1] for its details.

In the remainder of this section, we display the power of the transfer principle, applying it to produce a simple proof of a standard advanced calculus result which normally takes considerable development. While this can be found in any elementary treatise on non-standard analysis, it well illustrates "transfer".

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$.

Theorem 3.1. f is continuous at $x_0 \in D$ if and only if for every $x \in {}^*D$ for which $x \sim x_0$, ${}^*f(x) \sim {}^*f(x_0)$.

Proof. Suppose f is continuous at $x_0 \in D$. Let ε be an element of $\mathbb{R}^+$. There is, then, δ in $\mathbb{R}^+$ so that $\vDash (\forall x \in D)(|x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$. Transfer gives $^*\vDash (\forall x \in {}^*D)(|x-x_0| < \delta \Rightarrow |{}^*f(x) - {}^*f(x_0)| < \varepsilon)$. If now $x \in {}^*D$ and $x \sim x_0$ then $|x-x_0| < \delta$ for any δ in $\mathbb{R}^+$, hence $|{}^*f(x) - {}^*f(x_0)| < \varepsilon$ for any ε in $\mathbb{R}^+$. But then ${}^*f(x) \sim {}^*f(x_0)$ using Theorem 2.3 (3).

Suppose now that $x \sim x_0 \Rightarrow {}^*f(x) \sim {}^*f(x_0)$. Let $\varepsilon > 0$. For any infinitesimal δ in $\mathbb{R}^+$, if $|x-x_0| < \delta$ then $x \sim x_0$, so ${}^*f(x) \sim {}^*f(x_0)$ and it follows that $|{}^*f(x) - {}^*f(x_0)| < \varepsilon$ for any ε in $\mathbb{R}^+$. Fixing ε in $\mathbb{R}^+$, then

$$^*\vDash (\exists \delta \text{ in } {}^*\mathbb{R}^+)(x \in {}^*D), (|x-x_0| < \delta \Rightarrow |{}^*f(x_0 - {}^*f(x)| < \varepsilon).$$

Transferring this statement back to $\mathbb{R}$ gives the standard definition for "f is continuous at x_0".

While the above is sufficient to see how transfer operates, much more can be done from this point with very little effort. The proof of the next theorem is similar to the above so we omit it.

Theorem 3.2. f is uniformly continuous on $D \subset \mathbb{R}$ if and only if $\forall x,y$ in ${}^*D$ for which $x \sim y$, ${}^*f(x) \sim {}^*f(y)$.

The equivalence of continuity and uniform continuity on compact sets can now be shown without need of developing Bolzano-Weierstrass or Heine-Borel ideas.

Theorem 3.3. Let $f : [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$ then f is uniformly continuous on $[a,b]$.

Proof. Let x and y be elements of ${}^*[a,b]$ such that $x \sim y$. We need only show: ${}^*f(x) \sim {}^*f(y)$ by Theorem 3.2. Since x and y are in $\mathbb{F}$, there is $t \in \mathbb{R}$ so $t \sim x$; but since $x \sim y$ we have $t \sim y$ also by Theorem 2.3. Theorem 3.1 then gives us ${}^*f(t) \sim {}^*f(x)$ and ${}^*f(t) \sim {}^*f(y)$. The transitivity of " $\sim$ " (Theorem 2.3) again gives us that ${}^*f(x) \sim {}^*f(y)$ and the proof is complete.

We complete these theorems with a non-standard verification that $f(x) = \sin 1/x$ is not uniformly continuous on $(0,1]$.

Example. It suffices to find positive infinitesimals x and y so $^*\sin 1/x \not\sim {}^*\sin 1/y$, or equivalently, infinitely large positive numbers a and b so $^*\sin a \not\sim {}^*\sin b$. Letting ω be any positive infinite integer, $a = (\omega + 1/2)\pi$, $b = \omega\pi$ we have $^*\sin a = 1$, $^*\sin b = 0$ which does it. (That $^*\sin$ inherits its elementary evaluations from the sine function is a simple application of the transfer principle.)

While the above examples are well-known, we have produced a non-standard version of the most elemental monotonicity theorem to conclude this section. It avoids the traditional mean value theorem development. Note that the non-standard derivative is well-defined and behaves like a derivative merely from the transfer of the definition.

Lemma: Let $f : \mathbb{R} \to \mathbb{R}$. f is increasing on $\mathbb{R}$ if and only if for all $a \in \mathbb{R}$ and s,t in $\mu(a)$ such that $s < a < t$, $^*f(x) < {}^*f(t)$.

Proof. If f is increasing on $\mathbb{R}$, a straightforward transfer produces the conclusion.

Suppose f is not increasing on $\mathbb{R}$. A simple compactness argument produces $a \in \mathbb{R}$ so that

$$\vDash (\forall \varepsilon > 0)(\varepsilon \text{ in } \mathbb{R})(\exists\, s,t \text{ in } (a-\varepsilon,a+\varepsilon),\ s < a < t \text{ but } f(s) \geq f(t)).$$

Transfer of this statement produces a contradiction in the conclusion.

Theorem 3.4: Let $f : \mathbb{R} \to \mathbb{R}$ so that f' exists on $\mathbb{R}$ and $f'(a) > 0$ for all $a \in \mathbb{R}$. Then f is increasing on $\mathbb{R}$.

Proof. For any pair s,t in $\mu(a)$, $\frac{^*f(s)-{}^*f(a)}{s-a} \sim {}^*f'(a)$ and $\frac{^*f(t)-{}^*f(a)}{t-a} \sim {}^*f'(a)$. But then for $s < a < t$, both in $\mu(a)$

$$\frac{^*f(t)-{}^*f(s)}{t-s} = \frac{^*f(t)-{}^*f(a)}{t-a}\left[\frac{t-a}{t-s}\right] + \frac{^*f(s)-{}^*f(a)}{s-a}\left[\frac{a-s}{t-s}\right] \sim {}^*f'(a)\left[\frac{t-a}{t-s}\right] +$$

$^*f'(a)\left[\frac{a-s}{t-s}\right] = {}^*f'(a) > 0$, hence $^*f(s) < {}^*f(t)$.

Note: $s < a < t$ implies $\frac{a-s}{t-s}$ and $\frac{t-a}{t-s}$ are finite elements of $^*\mathbb{R}$, without which fact we do not have the infinitesimal approximation in the above line.

4. Superstructures and Universes: The Internal Set Distinction.

We begin by giving the details of the universal structures which we assumed existed in Section 3. Properties of these structures are stated without proofs since the proofs take considerable development and do not have much motivating power as far as analysis is concerned. Moreover, proofs can be found in standard references ([1], for example).

Let X be a non-empty set. The two cases of interest to us at this point are $\mathbb{R}$ and $^*\mathbb{R}$. In later sections, X will represent a topological space or a measure space. The objects of interest in analysis - subsets, functions, relations, etc. - can all be described within a framework of levels of sets as is well-known. A function on X, for example, is a set of subsets of subsets of X.

So, we create a superstructure for X to contain all these objects. $\mathcal{P}(X)$ will denote the power set of X. Let $X_1 = X \cup \mathcal{P}(X)$, $X_2 = X_1 \cup \mathcal{P}(X_1)$ and so on. Inductively we define X_n for each n. $\hat{X} = \bigcup_{n-1}^{\infty} X_n$ is called a superstructure of X; X is called the set of individuals of $\hat{X}$.

For any given non-empty set X, we can define a superstructure for S and for the set $^*S$ defined in Section 2. $\hat{S}$ and $^*\hat{S}$, however, bear no significant relation to each other: $^*\hat{S}$ is too large. We rectify this in such a way as to maintain a close parallel between the superstructure through the ultrafilter $\mathcal{F}$.

Recall from Section 2 that Z_0 is the set of functions from Γ to S, such that for f,g in Z_0, $f \sim g$ in case $f_\delta = g_\delta$ $\mathcal{F}$ - a.e. where $\mathcal{F}$ is the suitably chosen ultrafilter on Γ, and $^*S = \{\bar{f} : f \in Z_0\}$ where $\bar{f}$ represents an " ~ " equivalence class.

For any positive integer n, the set S_n is defined by $S_n = \mathcal{P}(S_{n-1}) \cup S_{n-1}$ as above. Let $Z_n = \{f : \Gamma \to \hat{S} : f_\delta \in S_n, \ \mathcal{F}\text{ - a.e.}\}$. Suppose that for given n, $\bar{f}$ has been defined for each $f \in Z_n$ and in such a way that $f_n \in ({}^*S)_n$. Then, for $f \in Z_{n+1} \setminus Z_n$ define

$$\bar{f} = \{\bar{g} : g \in Z_n \text{ and } g_\delta \in f_\delta \text{ a.e.}\}.$$

By induction, then, $\bar{f}$ has been defined for all $f \in Z = \bigcup_{n=1}^{\infty} Z_n$ and in such a way that $\bar{f} \in {}^*\hat{S}$. Finally ${}^*\tilde{S} = \{\bar{f} : f \in Z\}$ is called the <u>non-standard universe corresponding to</u> $\hat{S}$. S and ${}^*S$ are, respectively, the sets of individuals in the standard and non-standard universes.

Note what has occurred. The ultrafilter $\mathcal{F}$ has been used to select out of ${}^*\hat{S}$ certain elements and throw away the rest. This subset is ${}^*\tilde{S}$. It is this distinction which gives rise to the second key operating concept.

<u>Definition 4.1</u>. An element of ${}^*\tilde{S}$ is called an <u>internal set</u>. An element of ${}^*\hat{S} \setminus {}^*\tilde{S}$ is called an <u>external set</u>.

Speaking informally, internal sets are the objects we can make specific statements about by proper transferral from the standard universe. In the introduction of his book, [1], Martin Davis points out the contradiction in Leibniz's hope to create a set containing infinitesimals and having the same properties as $\mathbb{R}$.

Namely, such a set cannot have the same properties as $\mathbb{R}$ since, for example, it does not share with $\mathbb{R}$ the property of having infinitesimals. It is precisely this problem that the internal set distinction speaks to. It allows a calculus of infinitesimals but selects carefully those objects in the extension which "behave like" objects in the original space. Relative to the example given above, the set I of infinitesimals is not an internal set. Ways of determining which sets are internal await the careful distinction of language which we make in the next section.

5. Languages and the Star Operation.

The fundamental question concerning ${}^*S$ for our non-empty set S is to determine what can be said that is true. The language of mathematics, then, plays a major role.

Without going into tedious formalism, languages for each universe are constructed consisting initially of three types of objects:

(1) <u>fundamental logical symbols</u>: "element of", "negation", quantifiers, etc.

(2) <u>variables</u> to use in the formation of sentences

(3) A <u>copy of the universe</u> to serve as the substantive or <u>constant</u> elements of sentences.

The languages are then constructed inductively on the number of terms in a sentence. The two languages are developed in close parallel so as to have "isomorphic" semantics.

Let $\mathcal{L}$ denote the language of $\hat{S}$ and $^*\mathcal{L}$ the language of $^*\tilde{S}$. At this point we must begin to be more careful about the use of the star symbol " * ". As used in the previous sentence, it is merely part of a symbol denoting a set, so there is no problem. As used in the Rules and examples of Section 3, however, it names a specific mapping from $\hat{S}$ to $^*\tilde{S}$. We now clarify what this mapping is.

The first specification of "star" is with respect to languages. The second is with respect to sets of $\hat{S}$.

Let α be a mathematical statement. We use the symbol " $\vDash \alpha$ " to mean $\alpha \in \mathcal{L}$, or more simply, α is a true statement. Also, " $^* \vDash \beta$ " denotes: $\beta \in {}^*\mathcal{L}$.

<u>Definition 5.1</u>. If $\alpha \in \mathcal{L}$, then $^*\alpha \in {}^*\mathcal{L}$ is the statement obtained by replacing each constant b in α by its map $\bar{b}$ in $^*\tilde{S}$ (under the natural embedding of $\hat{S}$ in $^*\tilde{S}$, i.e., as the equivalence class of a constant function.)

Certain statements in a language are called <u>formulas</u>. The definition is complicated and corresponds with our intuitive idea of a formula so we spare the details.

For any universe U (the two of interest here being $\hat{S}$ and $^*\tilde{S}$), we denote $\mathcal{L}_U$ its language. " $U \vDash$ " signifies that a statement is true in U. We then have

<u>Definition 5.2</u>. Let $A \subset U$. A is said to be a <u>definable</u> set if there is a formula $\alpha = \alpha(x)$ in $\mathcal{L}_U$ so that

$A = \{b \in U : U \models A(b)\}$. α is the definition of A.

For example, all sets A in the standard universe $\hat{S}$ are definable by $A = \{b \in \hat{S} : b \in A\}$ ("$b \in A$" being a simple formula). This allows the definition of " $*$ " as a map between the standard and nonstandard universe.

A major theorem in nonstandard analysis guarantees that the definition of the set is independent of the specific formula which defines it. The proof can be found in [1] (page 28). With this in mind we make the following definition.

Definition 5.3. Let A be a set in $\hat{S}$ defined by the formula $\alpha \in \mathcal{L}$. So, ${}^*A = \{b \in \hat{S} : \models {}^*\alpha\ (b)\}$.

" $*$ " then defines a map from $\hat{S}$ to ${}^*\tilde{S}$. It is quite easy to show that for $x \in S$, ${}^*x = \bar{x}$ (cf. Section 4) and both are identified with x. What this mapping does is relate a set A in $\hat{S}$ to ${}^*A$ in ${}^*\tilde{S}$ through the language so that ${}^*A$ inherits many of the properties of A. The set ${}^*\mathbf{N}$, for example, behaves like the natural numbers in so far as its properties can be expressed through formulas.

Sets of the form ${}^*A$ for $A \in \hat{S}$ are called standard elements of ${}^*\tilde{S}$. So, the standard elements give us a large collection of internal sets in the nonstandard universe.

Moreover, " $*$ " is an isomorphism with respect to basic set operations (union, intersection, negation, containment, etc.); if f is a function from A to B, then ${}^*f$ is a function from ${}^*A$ to ${}^*B$. A very important property of internal sets is the following theorem. Its proof can be found in [1].

Theorem 5.1. If A is internal and $A \subset {}^*B$ for some $B \in \hat{S}$, then $A \in {}^*\mathcal{P}(B)$.

Applications can now be given which illuminate the role that internality plays. Recall that in Section 3, we proved that every nonempty standard (in present terminology) subset of ${}^*\mathbf{N}$ has a smallest element by simple appeal to the transfer principle. The above theorem allows an easy extension from which we draw useful consequences.

Theorem 5.2. Every nonempty internal subset of ${}^*\mathbb{N}$ has a smallest element.

Proof. Transfer the statement

$$\vdash (\forall X \in \mathcal{P}(\mathbb{N}))(X \neq \emptyset)(\exists m \in X)(\forall x \in X)(m \leq x) \quad \text{and}$$

then apply Theorem 5.1.

Theorem 5.3. If $w \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $n \in \mathbb{N}$, then $n < w$.

Proof. Suppose $w \leq n$ for some n and assume this n is the smallest such. Then $n-1 < w < n$. But since no standard integer lies between two consecutive integers, neither can a nonstandard integer (a simple transfer argument), and so, a contradiction.

Theorem 5.4. ${}^*\mathbb{N} \setminus \mathbb{N}$ is an external set.

Proof. If ${}^*\mathbb{N} \setminus \mathbb{N}$ is internal, it has a smallest element w_0 by 5.2. For any $n \in \mathbb{N}$, $w_0 > n+1$, hence $w_0-1 > n$ for each $n \in \mathbb{N}$, i.e., $w_0-1 \in {}^*\mathbb{N} \setminus \mathbb{N}$, a clear contradiction.

As a corollary of the above, we obtain a simple proof of the intermediate value theorem which avoids the Bolzano-Weierstrass development but captures the spirit of Bolzano's original proof.

Theorem 5.5. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $f(0) < 0$ and $f(1) > 0$. Then there is $c \in [0,1]$ so that $f(c) = 0$.

Proof. Let $w \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $A = \{n \in {}^*\mathbb{N} : {}^*f(\frac{n}{w}) \geq 0 \text{ and } 0 \leq \frac{n}{w} \leq 1\}$. A is a non-empty internal subset (reader may supply argument), hence has a smallest element n_0. So, there is $n_0 \in {}^*\mathbb{N}$ such that ${}^*f(\frac{n_0}{w}) \geq 0$, but ${}^*f(\frac{n_0-1}{w}) \leq 0$. From Section 2, since $\frac{n_0}{w} \in \mathbb{F}$, $\exists\, c \in \mathbb{R}$ so $c \sim \frac{n_0}{w}$. But $\frac{n_0}{w} \sim \frac{n_0-1}{w}$ implies $c \sim \frac{n_0-1}{w}$. By continuity, then $f(c) \sim {}^*f(\frac{n_0}{w}) \geq 0$,

$f(c) \sim {}^*f(\frac{n_0-1}{w}) < 0 \Rightarrow f(c) = 0.$

We conclude by noting that internal sets need not be standard. This example also provides the pattern for showing the internality of A in the above proof.

Let $w \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $B = \{n \in {}^*\mathbb{N} : n \geq w\}$. Transferring the statement

$$\vdash (\forall m \in \mathbb{N})(\{n \in \mathbb{N} : n \geq m\} \in \mathcal{P}(\mathbb{N})), \quad \text{gives} \quad B \in {}^*\mathcal{P}(\mathbb{N}),$$

hence B is internal by 5.1. Clearly, $B \neq {}^*C$ for any $C \subset \mathbb{N}$ so B is not standard.

6. New Applications: Locally Valid Suborderings.

Theorems and examples in previous sections are variations of relatively standard aspects (pun intended) of the elementary treatment of nonstandard analysis. We now apply these ideas to more recent research in real analysis and generate results not previously published.

In [2], Leinfelder introduces the concept of "locally valid subordering" as a unifying principle to recast several fundamental theorems in real analysis. Simple proofs of these theorems are then given which avoid direct Bolzano-Weierstrass and Heine-Borel constructions.

A nonstandard approach leads to a unifying principle which underplays the order relation and emphasizes the "localization" of properties which occurs when dealing with compact sets. More usefully, the nonstandard approach generalizes easily to arbitrary metric spaces as we shall see in a later section and places several of Leinfelder's examples in more general context. Since the generalization uses the third main tool of nonstandard analysis, the concurrence principle, our attention is restricted to $\mathbb{R}$ in this section.

Let $I_0 = [a,b] \subset \mathbb{R}$ be a compact interval and $\mathcal{J}$ some class of compact subintervals of I_0. Let $P : \mathcal{J} \to \mathcal{L}$ denote some property of compact subintervals of I_0. P(J) is said to be valid if the property holds on J. Also P is an additive property if whenever $J = J_1 \cup J_2$, with J, J_1 and J_2 in $\mathcal{J}$, then $P(J_1)$ and $P(J_2)$ valid implies P(J) is valid. A subset $K \subset {}^*\mathbb{R}$ is infinitesimal in case k_1, k_2 in K implies $k_1 \sim k_2$. This sets up the principle condition needed.

<u>Definition 6.1</u>. The property P is <u>infinitesimally valid</u> on $\mathcal{J}$ in case P is additive and $^*P(K)$ holds for all $K \in {}^*\mathcal{J}$ for which K is infinitesimal.

Note that simple uses of transfer give us the facts that if P is additive on $\mathcal{J}$, then $^*P$ is additive on $^*\mathcal{J}$ and that $^*\mathcal{J}$ is a collection of subintervals $[z,w]$ of $^*I_0$.

<u>Example 6.1</u>. Let $f : I_0 \to \mathbb{R}$ be differentiable and such that $f'(x) > 0$ for $x \in I_0$. Fix $y \in I_0$, $y \neq a$. Let $\mathcal{J} = \{J = [x,y], x \in I_0, x < y\}$. Let P(J) be the statement:

$$\vdash (J = [x,y]) \left(\frac{f(y)-f(x)}{y-x} > 0\right),$$ for each $J \in \mathcal{J}$, P is easily shown to be additive.

$^*P(K)$ is: $^* \vdash (K = [z,y])\left(\frac{^*f(y)-{}^*f(z)}{y-z} > 0\right)$, for each $K \in {}^*\mathcal{J}$. If K is infinitesimal, $\frac{^*f(y)-{}^*f(z)}{y-z} \sim {}^*f'(y) > 0$, hence P is an infinitesimally valid property on $\mathcal{J}$.

<u>Example 6.2</u>. Let $\{\mathcal{O}_\alpha : \alpha \in \mathfrak{a}\}$ be an open cover of I_0, $\mathcal{J}$ the class of closed subintervals of I_0 and P(J) the property: some finite subset of $\{\mathcal{O}_\alpha : \alpha \in \mathfrak{a}\}$ covers J. P is clearly additive. $^*P(K)$ states that every $K \in {}^*\mathcal{J}$ has a *-finite subcover from $^*\{\mathcal{O}_\alpha : \alpha \in \mathfrak{a}\}$. If K is infinitesimal, then K is a subset of some monad $\mu(a)$, $a \in I_0$. But since $a \in \mathcal{O}_\alpha$ for some $\alpha \in \mathfrak{a}$, $\mu(a) \subset {}^*\mathcal{O}_\alpha$, hence K is covered by the *-finite subset of one element of $^*\{\mathcal{O}_\alpha : \alpha \in \mathfrak{a}\}$. So, P is an infinitesimally valid property on $\mathcal{J}$.

<u>Example 6.3</u>. Let f be a regulated function on I_0, i.e., one having finite left and right limits $f(c-)$ and $f(c+)$ for each $c \in (a,b)$ with $f(a+)$, $f(b-)$ existing also. Let $\mathcal{J}$ be the class of all closed subintervals of I_0 and, for fixed $n \in \mathbb{N}$, let P(J) for $J \in \mathcal{J}$ denote the property: "there is a step function $\varphi(x)$ defined on J such that $\varphi(x) \geq f(x)$ and $\varphi(x) - f(x) < 1/n$ for all $x \in J$". P is clearly additive. If $K \in {}^*\mathcal{J}$ is infinitesimal, let $c \in I_0$ be such that $K \subset \mu(c)$. Define $\psi(x)$ as follows:

$$\psi(x) = \begin{cases} f(c-) + \frac{1}{2n} & \text{for } x \in K, \ x \leq c \\ f(c+) + \frac{1}{2n} & \text{for } x \in K, \ x > c \end{cases}$$

ψ is easily shown to be an element of the transferred set of step functions on closed subintervals of I_0. In short, P is an infinitesimally valid property on $\mathcal{J}$.

In like fashion, Leinfelder's other examples can be rewritten as infinitesimally valid properties; we will not give the details here. Rather, in the spirit of [2], we prove a general theorem which we use to extract well-known results from the above examples.

Theorem 6.1. If P is an infinitesimally valid property on $\mathcal{J}$, then P(J) is valid for all $J \in \mathcal{J}$.

Proof. Assume the conclusion is false; so there is $J = [x,y] \in \mathcal{J}$ such that P(J) is not valid. Let $w \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $z_n = (1 - \frac{n}{w})x + \frac{n}{w} y$, for any $n \leq w$, $n \in {}^*\mathbb{N}$. Note: $z_n \in {}^*J$. Let $A = \{n \in {}^*\mathbb{N} : n \leq w, \ {}^*P([x,z_n]) \text{ is not valid}\}$. A is a nonempty internal subset of ${}^*\mathbb{N}$, hence by Theorem 5.2, A has a smallest element n_0. So, ${}^*P([x,z_{n_0}])$ is not valid but ${}^*P([x,z_{n_0-1}])$ is valid. By the additivity of ${}^*P$, ${}^*P([z_{n_0-1},z_{n_0}])$ is not valid. But $[z_{n_0-1},z_{n_0}]$ is infinitesimal, contradicting the hypothesis, thus completing the proof.

We note now, in light of this theorem, that Example 6.1 implies the fundamental monotonicity theorem, Example 6.2 the Heine-Borel theorem and Examples 6.3 the uniform approximation of regulated functions by step functions.

7. Concurrent Relations.

The concurrence principle is the third fundamental tool needed to do non-standard analysis and gives the subject much of its power.

Let r be a relation in $\hat{S}$, (a,b) denote an ordered pair and dom(r) the domain of r.

Definition 7.1. r is said to be concurrent in $\hat{S}$ in case whenever for some $n \in N$, $\{a_1,\dots,a_n\} \subset dom(r)$, there is an element $a \in \hat{S}$ such that $(a_k,a) \in r$, $k = 1,\dots,n$.

The principal result can now be stated. Its proof is also a major piece of work in nonstandard analysis and so will only be referenced. ([1], page 34.) A careful choice of the index set Γ and the ultrafilter need be made and such choices can be made for all traditional analytic settings. One of the main reasons for the careful selection of the nonstandard universe $^*\tilde{S}$ from $^*\hat{S}$ is that in the former, the result below is valid. In Abraham Robinson's terminology, $^*S$ is then called an enlargement of S.

Theorem 7.1. (The Concurrence Principle) Let r be a concurrent relation in $\hat{S}$. Then, there is an element $b \in {}^*\tilde{S}$ such that $({}^*a,b) \in {}^*r$ for all $a \in dom(r)$. (We will abbreviate this theorem as CT.)

What is added to the theory at this point is the existence of universal elements under fairly general hypotheses. An elementary application of CT, for example, provides a cleaner proof of the existence of elements of $^*N \setminus N$, i.e., infinite integers.

Let $r = \{(x,y),\ x \in N,\ y \in N,\ x < y\}$. r is clearly a concurrent relation. By CT, then, there is an element $z \in {}^*N$ so that $x < z$ for all $x \in N$. z is necessarily in $^*N \setminus N$.

The next section will provide more substantial applications.

8. Metric and Topological Spaces: Extensions of Infinitesimally Valid Properties.

We now assume $(X,\mathcal{T})$ is a topological space where X is an infinite set and $\mathcal{T}$ a topology on X. Let $^*X$ denote a suitably chosen enlargement of X. By this we ensure that $^*\tilde{X}$ is defined as in the previous sections (where it had the general name $^*\tilde{S}$) so that transfer, concurrence and internality behave as described in general. We recall also the natural embedding of X in $^*X$ so we regard it as a subset. Let $\mathcal{T}_p$ denote the set of open neighborhoods of the point $p \in X$. Then

Definition 8.1. $\mu(p) = \cap \{{}^*G : G \in \mathcal{T}_p\}$ is called the monad of p and we write $q \sim p$ to denote $q \in \mu(p)$.

This gives a type of "canonical infinitesimal neighborhood" of p. For example, the set I of infinitesimals in ${}^*\mathbb{R}$ is the monad of zero; monads are then not generally internal sets. (Note: $q \sim p$ is not, as defined, a symmetric relation.) $\mu(p)$ does, however, contain an internal "open neighborhood" of p.

Theorem 8.1. $\forall\ p \in X$, there is an internal set $D \in {}^*\mathcal{T}_p$ so that $D \subset \mu(p)$.

To prove this, let $r = \{(A,B) : A$ and B are in $\mathcal{T}_p$, $A \supset B\}$ and apply the Concurrence principle to the relation r.

The following theorems are fundamental to the study of nonstandard topological spaces. Their proofs are elementary enough to be supplied by the reader (liberally applying Theorem 8.1) or they can be found in [1].

Theorem 8.2. X is Hausdorff if and only if for each pair p,q in X, $p \neq q$, $\mu(p) \cap \mu(q) = \phi$.

Theorem 8.3. G is open in X if and only if for each $p \in G$, $\mu(p) \subset {}^*G$.

Theorem 8.4. F is closed in X if and only if for all $p \in X$, $\mu(p) \cap {}^*F \neq \phi \Rightarrow p \in F$.

Definition 8.1. $q \in {}^*X$ is near-standard if there is $p \in X$ so $q \sim p$. (p is called the standard part of q and we write $p = st(q)$).

Theorem 8.5. $K \subset X$ is compact if and only if for each $q \in {}^*K$, there is $p \in K$ so $q \sim p$.

If X is a metric space, several advantages accrue. For $p \in X$ it is easily shown that $\mu(p) = \{q \in {}^*X : {}^*d(p,q) < \varepsilon,\ \forall\ \varepsilon$ in $\mathbb{R}^+\}$ where d is the metric on X. Define $p \sim q$ in $^*X$ to mean $^*d(p,q) < \varepsilon$ for all real $\varepsilon > 0$, we extend the defnition and make it symmetric.

For the remainder of this section, X denotes a metric space. In this setting, we generalize the "infinitesimally valid" ideas of Section 6 and give some examples.

Let $K_0 \subset X$ denote a compact set and $\mathcal{K}$ some subclass of the compact subsets of K_0. Using our canonical symbols for languages, universes, etc., we let $P : \mathcal{K} \to \mathcal{L}$ be some property of sets in $\mathcal{K}$. P(K) is <u>valid</u> if it is true. P is <u>additive</u> on $\mathcal{K}$ if whenever K, K_1, K_2 are in $\mathcal{K}$, $K = K_1 \cup K_2$ and $P(K_1)$ and $P(K_2)$ are both valid, then P(K) is valid. A subset L of $^*K_0$ is <u>infinitesimal</u> in case $x \sim y$ for all x,y in L.

<u>Definition 8.2</u>. P is said to be <u>infinitesimally valid</u> in case P is additive and $^*P(L)$ is valid for all $L \in {}^*\mathcal{K}$ for which L is infinitesimal.

We note for use in the examples that the nonstandard definition of continuity given previously generalizes. Namely f is continuous at p in case for q in $^*X$, $q \sim p$ we have $^*f(q) \sim {}^*f(p)$.

<u>Example 7.1</u>. Let $f : K_0 \to \mathbb{R}$ be continuous on the compact subset K_0 of the metric space X. Suppose f does not attain a maximum on K_0. Let $\mathcal{K}$ be the compact subsets of K_0 and P(K) defined by: there is $m \in K_0$ so $f(k) < f(m)$ for all $k \in K$. P is obviously additive. Let L be an infinitesimal subset of $^*\mathcal{K}$. By Theorem 8.5, there is $c \in K_0$ so that $y \sim c$ for all $y \in L$, hence $^*f(y) < {}^*f(m)$ for all $y \in L$, i.e., $^*P(L)$ is valid, hence P is infinitesimally valid.

<u>Example 7.2</u>. With f, K_0 and $\mathcal{K}$ as in Example 7.1, let $\varepsilon > 0$ and P(K) be the statement: there is $\delta > 0$ so that if $\operatorname{diam}(K) < \delta$ then $|f(x)-f(y)| < \varepsilon$ for all x,y in K. If L is an infinitesimal subset of $^*\mathcal{K}$ then any δ suffices since z,w in $^*L \Rightarrow \exists\ c \in K_0$ so $z \sim c \sim w$, hence $^*f(z) \sim {}^*f(c)$ and $^*f(w) \sim {}^*f(c)$, implying $^*f(z) \sim {}^*f(w)$. So $^*P(L)$ is infinitesimally valid.

While other examples could be given, the above two give the spirit of the generalization. So, we state and prove the unifying principle. The central tool in the proof is the Concurrence Principle.

Theorem 8.6. Let K_0 be a compact subset of the metric space X, and $\mathcal{K}$ the class of compact subsets of K_0. If P is infinitesimally valid, then P is valid on $\mathcal{K}$.

Proof. Suppose there exists a $K \in \mathcal{K}$ so $P(K)$ is not valid. Covering K with the interiors of a finite set of closed spheres of radius 1, one finds a compact subset $K_1 \subset K$ so $\mathrm{diam}(K_1) < \mathrm{diam}(K)$ and $P(K_1)$ is not valid using the additivity of P. In like manner we choose a nested sequence of $K_n \in \mathcal{K}$ so $P(K_n)$ is not valid and $\mathrm{diam}(K_n) < 1/2^n \,\mathrm{diam}(K)$. Define the relation r with domain the sequence $\{K_n : n \in \mathbb{N}\}$ by $(K,J) \in r$ in case $K = K_n$ for some $n \in \mathbb{N}$, $J \in \mathcal{K}$, $J \subset K$ and $P(J)$ is not valid. r is concurrent since if $K_{n_1},\ldots,K_{n_m}$ are chosen from $\mathrm{dom}(r)$, $J = K_p$, $p = \max\{n_1,\ldots,n_m\}$ satisfies $(K_{n_i},J) \in r$, $i = 1,\ldots,m$. By CT, there is $L \in {}^*\mathcal{K}$ so ${}^*P(L)$ is not valid and $L \subset {}^*K_n$ for all $n \in \mathbb{N}$. But then L is infinitesimal, contradicting the hypothesis.

The contradiction inherent in $P(K_0)$ being valid in Exercise 7.1 gives the maximum theorem for compact sets in metric spaces. The validity of P in Exercise 7.2 is the uniform continuity theorem.

9. Saturation: Application to Nonstandard Path Continuity.

The universal maximizing (or minimizing) elements supplied by the Concurrence Theorem are not sufficient for certain applications in analysis, due principally to the domain restrictions of concurrence.

Letting S be an infinite set, $\hat{S}$ and ${}^*\tilde{S}$ the structure defined previously, we make the following definition.

Let $\mathcal{K}$ be an infinite cardinal.

Definition 9.1. ${}^*\tilde{S}$ is said to be a $\mathcal{K}$-saturated in case for any internal relation ρ in ${}^*\tilde{S}$ that is concurrent on a subset A of its domain (i.e., $\forall\, n \in \mathbb{N}$, if $\{x_1,\ldots,x_n\} \subset A$, $\exists\, y \in \mathrm{ran}(\rho)$ so that $\{(x_1,y),\ldots,(x_n,y)\} \subset A$), and $\mathrm{card}(A) < \mathcal{K}$, then there exists $y^* \in {}^*\tilde{S} \cap \mathrm{ran}(\rho)$ so that $(x,y^*) \in \rho$ for all $x \in A$.

By careful choice of the ultrafilter, κ-saturated models can be constructed. By choosing κ so card(S) $< \kappa$, the Concurrence Principle is guaranteed. We assume that κ is always large enough.

Saturation is more powerful than concurrence in two regards. First, the binary relation need only be internal in $^*\tilde{S}$ rather than standard in $\hat{S}$. Second, the set A can be arbitrary (within cardinality restriction) whereas it must be standard in the Concurrence Principle. For greater elucidation of the foundations of saturation, we refer the reader to [4], (page 27 ff.).

Wattenberg used saturation to give a nonstandard characterization of approximate continuity [6]. Letting $\mathfrak{M}$ denote a higher order structure including $\mathbb{R}$ and its Lebesgue measure space, and $^*\mathfrak{M}$ a κ-saturated nonstandard extension of $\mathfrak{M}$ with κ sufficiently large he made the following definition on $[0,1]$.

Definition 9.2. $x \in {}^*[0,1]$ is negligible in case there is a standard set $A \subset [0,1]$ so that

(i) st(x) is a point of dispersion for A

(ii) $x \in {}^*A$.

Let $\mathfrak{N}$ denote the set of negligible points of $^*[0,1]$. Then,

Theorem 9.1. Suppose $f : [0,1] \to \mathbb{R}$ and $x \in [0,1]$. Then f is approximately continuous at x if and only if for each $t \in \mu(x) \setminus \mathfrak{N}$, $^*f(t) \sim {}^*f(x)$.

In some sense, then, the set $\mathfrak{N}$ universally selects from each monad the points eliminated from consideration by the local selection of a set of density 1 at each of the points of $[0,1]$.

Proofs are not supplied since an examination of the result in [6] indicates that it holds more generally in the setting of continuity paths described by Bruckner, O'Malley and Thomson in [5].

Definition 9.3. Let $x \in \mathbb{R}$

(1) A path leading to x is a set $E_x \subset \mathbb{R}$ such that $x \in E_x$ and x is an accumulation point of E_x.

(2) A <u>system of paths at x</u>, $\mathcal{E}_x$, is a collection of paths leading to x.

(3) A <u>system of paths</u> is a collection $\mathcal{E} = \cup \{\mathcal{E}_x : x \in \mathbb{R}\}$.

The properties listed below are easily shown to hold for several of the well-known systems of paths studied in [5], including the ordinary type, the (1,1)-density type and the qualitative type. They are sufficient to give us the generalization referred to above. The (1,1)-density type is the result in [6].

Definition 9.4. A system of paths will be called locally thick in case the following conditions are satisfied.

(i) If $E_x^{(1)}$ and $E_x^{(2)}$ are in $\mathcal{E}_x$, there is $E_x^{(3)}$ in $\mathcal{E}_x$ so that $E_x^{(3)} \subset E_x^{(1)} \cap E_x^{(2)}$.

(ii) If $E_x \in \mathcal{E}_x$, then for $\eta > 0$, $E_x \cap (x-\eta, x+\eta) \in \mathcal{E}_x$.

(iii) If $E_x^n \in \mathcal{E}_x$ for $n \in \mathbb{N}$, there is decreasing sequence $\langle c_n \rangle \to 0$ so that $\bigcup_{n=1}^{\infty} (J_n \cap E_x^n) \cup \{x\} \in \mathcal{E}_x$ where $J_n = (x-c_n, x-c_{n+1}] \cup [x+c_{n+1}, x+c_n)$.

Definition 9.5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be $\mathcal{E}$-continuous at x in case $\exists\ E_x \in \mathcal{E}_x$ so that $\lim_{\substack{y \to x \\ y \in E_x}} f(y) = f(x)$.

For the remainder of this section, the family $\mathcal{E}$ will be assumed to be locally thick. Also, D^c denotes the complement of D in $\mathbb{R}$.

Theorem 9.2. $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{E}$-continuous if and only if for each $\varepsilon > 0$ $\{y : |f(y) - f(x)| \geq \varepsilon\} \subset E_x^c$ for some $E_x \in \mathcal{E}_x$.

Proof. Suppose f is $\mathcal{E}$-continuous at x. Then, there is $E_x \in \mathcal{E}_x$ so that: for each $\varepsilon > 0$ there is $\delta > 0$ so that $\{y : |f(y) - f(x)| \geq \varepsilon\} \subset [E_x \cup (x-\delta, x+\delta)]^c$. Property (ii) of 9.4 produces $F_x \in \mathcal{E}_x$ so $\{y : |f(y) - f(x)| \geq \varepsilon\} \subset F_x^c$.

Suppose for $\varepsilon > 0$, $\{y : |f(y) - f(x)| \geq \varepsilon\} \subset E_x^c$. Given $n \in \mathbb{N}$, there is $E_x^n \in \mathcal{E}_x$ so $E_x^n \subset \{y : |f(y) - f(x)| < 1/n\}$. By Property (iii) of 9.4, there is $\langle c_n \rangle \to 0$ so $F_x = (E_x^n \cap J_n) \cup \{x\} \in \mathcal{E}_x$. It is easily computed that

$$\lim_{\substack{y \to x \\ y \in F_x}} f(y) = f(x).$$

For the system of paths $\mathcal{E}$ we define the nonstandard set $\mathfrak{N}(\mathcal{E})$ in ${}^*\mathbf{R}$.

<u>Definition 9.6</u>. $\mathfrak{N}(\mathcal{E}) = \{z \in {}^*\mathbf{R} : \exists\ x \in \mathbf{R}$ and $E_x \in \mathcal{E}_x$ so that $x = st(z)$ and $z \in {}^*E_x{}^c\}$. $\mathfrak{N}(\mathcal{E})$ is called the <u>$\mathcal{E}$-negligible</u> elements of ${}^*\mathbf{R}$.

<u>Theorem 9.3</u>. $f : \mathbf{R} \to \mathbf{R}$ is $\mathcal{E}$-continuous at x if and only if for every $y \in \mu(x) \setminus \mathfrak{N}(\mathcal{E})$, ${}^*f(y) \sim {}^*f(x)$.

The proof is by way of the following lemma.

<u>Lemma</u>. Suppose $A \subset \mathbf{R}$ and $x \in \mathbf{R}$. Then $A \subset E_x{}^c$ for some $E_x \in \mathcal{E}_x$ $\iff$ $\mu(x) \cap {}^*A \subset \mathfrak{N}(\mathcal{E})$.

<u>Proof</u>. Suppose first that A is not a subset of $E_x{}^c$ for any $E_x \in \mathcal{E}_x$. By Property (ii), for $\eta > 0$ and any $E_x \in \mathcal{E}_x$, $A \cap E_x \cap (x-\eta,x+\eta) \neq \phi$. The transfer principle gives us ${}^*A \cap {}^*E_x \cap {}^*(x-\rho,x+\rho) \neq \phi$ for ρ infinitesimal, and hence ${}^*A \cap {}^*E_x \cap \mu(x) \neq \phi$. Let r be the relation defined on $\mathcal{E}_x \times \mu(x)$ by $(E_x,y) \in r$ in case $E_x \in \mathcal{E}_x$ and $y \in {}^*E_x \cap {}^*A \cap \mu(x)$. We show that r is a concurrent, nonstandard relation. If $\{E_x{}^1,\ldots,E_x{}^n\}$ is a finite subset of $\mathcal{E}_x$, there is by Property (i) an $E_x \subset \bigcap_{k=1}^{n} E_x{}^k$. Pick $y \in {}^*E_x \cap {}^*A \cap \mu(x)$. Then $(E_x{}^k,y) \in r$ for $k = 1,\ldots,n$ and the concurrence of r is established. The saturation of our model now gives a $z \in \mu(x)$ so that $(E_x,z) \in r$ for all $E_x \in \mathcal{E}_x$. But then $z \in \mu(x) \cap {}^*A$ but $z \notin E_x{}^c$ for any $E_x \in \mathcal{E}_x$. So, $z \notin \mathfrak{N}(\mathcal{E})$ and the conclusion is negated.

Assume now $A \subset E_x{}^c$ for some $E_x \in \mathcal{E}_x$. Let $z \in \mu(x) \cap {}^*A$. Then $x = st(z)$ and $z \in {}^*E_x{}^c$, hence $z \in \mathfrak{N}(\mathcal{E})$. So $\mu(x) \cap {}^*A \subset \mathfrak{N}(\mathcal{E})$ and the proof of the lemma is complete.

<u>Proof of Theorem</u>.

By Theorem 9.2, f is $\mathcal{E}$-continuous at $x \iff \forall\ \varepsilon > 0$, $\{y : |f(y) - f(x)|$

$\geq \varepsilon\} \subset E_x^c$ for some $E_x \in \mathcal{E}_x$ <=> $\forall\ \varepsilon > 0$, $\mu(x) \cap \{y : |{}^*f(y) - {}^*f(x)| \geq \varepsilon\} \subset \mathfrak{N}(\mathcal{E})$ (by above lemma). We need only show this latter statement is equivalent to the conclusion of the theorem.

Assume first the latter statement and let $y \in \mu(x) \sim \mathfrak{N}(\mathcal{E})$. Then for each real $\varepsilon > 0$, $|{}^*f(y) - {}^*f(x)| < \varepsilon$, hence ${}^*f(y) \sim {}^*f(x)$.

On the other hand, assume the conclusion of the theorem. For any $\varepsilon > 0$, let $z \in \mu(x) \cap \{y : |{}^*f(y) - {}^*f(x)| \geq \varepsilon\}$. If $z \notin \mathfrak{N}(\mathcal{E})$ then ${}^*f(z) \sim {}^*f(x)$, contradicting the choice of z. So $z \in \mathfrak{N}(\mathcal{E})$, verifying the latter statement.

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