Real Analysis Exchange Vol. 10 (1984-85)

Ryszard Jerzy Pawlak, Institute of Mathematics, Lodz University, Banacha 22, 90-238 Lodz, POLAND.

ON SOME RINGS OF SWIATKOWSKI FUNCTIONS

In 1977, T. Mank and T. Swiatkowski in paper [1] defined a new class of functions. According to the terminology adopted in [2] elements of this class we call Swiatkowski functions.

Defintion. We say that $f:R \to R$ is a Swiatkowski function if for every two points $x,y \in R$ such that $f(x) \neq f(y)$ there exists a point z of continuity of f such that $z \in (x,y)$ and $f(z) \in (f(x),f(y))$.

We assume the notation (a,b) in either case $a < b$ or $b < a$.

Let C_f (D_f) denote the set of all continuity (discontinuity) points of f.

It is known that there exist Swiatkowski functions f and g such that $f + g$ is not a Swiatkowski function. So the question whether it is possible to form a ring of Swiatkowski functions, containing all continuous functions and a fixed Swiatkowski function f, seems to be interesting.

For a Swiatkowski function $f:R \to R$ let $RS(f)$ denote the class of all complete rings K of Swiatkowski functions such that $f \in K$ and $C \subset K$, where C denotes the class of all continuous functions. (A ring K of real functions is complete if for every $g \in K$, $|g|$ also belongs to K.)

Now the above question can be formulated in the following way: Under what hypothesis on f is $RS(f) \neq \emptyset$?

First we consider the simple case of $D_f = \{x_o\}$.

Theorem 1. Let f be a Swiatkowski function such that $D_f = \{x_o\}$. Then $RS(f) \neq \emptyset$ if and only if x_o is a Darboux point of f.

The next theorem gives the answer to the above problem in a general case. In light of Theorem 1 we add the additional assumption that the functions under consideration are Darboux functions.

Theorem 2. Let f be a Darboux, Swiatkowski function in Baire class 1. Then $RS(f) \neq \emptyset$.

It is possible to construct an example of a nonmeasurable Swiatkowski function such that $RS(f) \neq \emptyset$. This fact follows from the next theorem.

Theorem 3. Let f be a Darboux function such that the set D_f is a nowhere dense set and let f fulfill the following condition: for every point $x \in D_f$ and every $\eta > 0$ there exists $\delta(x,\eta) > 0$ such that if S is a component of C_f and $\rho(x,S) < \delta(x,\eta)$, then $\rho(f(x),f(S)) < \eta$. Then f is a Swiatkowski function and moreover $RS(f) \neq \emptyset$. ($\rho(x,A) = \inf_{a \in A} |x-a|$).

Theorem 3 is proved by constructing a topology O such that a ring of real functions continuous in the topology O belongs to $RS(f)$. Moreover every real function f continuous in the topology O is a Darboux function. (See the proof of Theorem 1 in [3].)

With regard to the above remarks we can formulate the following open problems.

<u>Problem 1</u>. Characterize the Swiatkowski functions f such that $RS(f) \neq \emptyset$.

<u>Problem 2</u>. Assuming that for some Swiatkowski function f $RS(f) \neq \emptyset$, characterize the functions g belonging to some ring $K \in RS(f)$.

<u>Remark</u>. The analogously questions can also be asked for Darboux functions. (See [3].)

REFERENCES

[1] Mank, T. and Swiatkowski, T., "On some class of functions with Darboux's characteristic", Zesz. Nauk. P.L .Mat.z.11, 1977.

[2] Pawlak, H. and Wilczynski, W., "On the condition of Darboux and Swiatkowski for functions of two variables", Zesz. Nauk. P.L. Mat.z.15, 1982, pp. 31-35.

[3] Pawlak, R.J., "On rings of Darboux functions", Colloquium Math. (to appear).

Received January 29, 1985