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# Convergence in Length and Area of the Bernstein and Kantorovitch Polynomials

This is a survey of the Ph.D. thesis of John Joseph Loughlin done under my guidance at Purdue University in 1971.

Let $B_k^f(x)$ be the k-th Bernstein polynomial on the n-cube $I^n$, defined to be

$$B_k^f(x) = \sum_{\nu_1=0}^{k} \cdots \sum_{\nu_n=0}^{k} p_{\nu_1 k}(x_1)\cdots p_{\nu_n k}(x_n) f\left(\frac{\nu_1}{k}, \ldots, \frac{\nu_n}{k}\right),$$

where

$$p_{\nu k}(x) = \binom{k}{\nu} x^\nu (1-x)^{k-\nu} \quad \text{for} \quad x \in [0,1].$$

The k-th Kantorovitch polynomial is defined to be

$$P_k^f(x) = \sum_{\nu_1=0}^{k} \cdots \sum_{\nu_n=0}^{k} p_{\nu_1 k}(x_1)\cdots p_{\nu_n k}(x_n)\cdot(k+1)^n \int_{\frac{\nu_1}{k+1}}^{\frac{\nu_1+1}{k+1}} \cdots \int_{\frac{\nu_n}{k+1}}^{\frac{\nu_n+1}{k+1}} f(s)ds.$$

For any $n$, for continuous $f$, $B_k^f$ converges uniformly on $I^n$ to $f$. For $n = 1$, the variations and lengths of the $B_k^f$ converge, respectively, to the variation and length of $f$.

For $n > 1$, the Tonelli variations and areas are considered. The definition of BVT, bounded variation in the sense of Tonelli, and ACT, absolute continuity in the sense of Tonelli, as well as that of the surface area $A(f)$ of the surface given by the continuous function $f$ are found in Saks for $n = 2$. The extensions to $n > 2$ are the natural ones.

A continuous $f: I^n \to R^1$ is in ACT if it is absolutely continuous in each variable for almost all values of the other $n - 1$ variables and if all the first partial derivatives are in $L_1$. Thus $f \in \text{ACT}$ if and only if $f$ is continuous and belongs to the Sobolev space $W_1^1$. If $\frac{\partial f}{\partial x_i} \in L_p$, $p > 1$, then $f \in W_p^1$.

Suppose $f \in \text{ACT}$. The variations

$$\Phi_i(f) = \int_{I^n} \frac{\partial f}{\partial x_i}\, dx, \ i = 1,2,\ldots,n \ \text{ are finite.}$$

Suppose $\{f_k\}$ converges to $f$ in $L_1$. We say that $\{f_k\}$ converges to $f$ <u>in variation</u> if $\lim_{k \to \infty} \Phi_i(f_k) = \Phi_i(f)$, $i = 1,\ldots,n$. If $\lim_{k \to \infty} \Phi_i(f - f_k) = 0$, $i = 1,\ldots,n$, we say that $\{f_k\}$ converges to $f$ in <u>strong variation</u>. If $\lim_{k \to \infty} A(f_k) = A(f)$, we say that $\{f_k\}$ converges to $f$ in <u>area</u>.

In one dimension it is classical that if $f$ is continuous on $[0,1]$ then $\Phi(B_k^f) \leq \Phi(f)$, $k = 1,2,\ldots$. Using this fact, it readily follows that $\{B_k^f\}$ converges to $f$ strongly in variation, and from this that it converges both in variation and in length.

The above inequality does not hold for $n > 1$. Indeed, the result just stated for $n = 1$ now holds for $f \in W_p^1$, $p > n$, but fails to hold for some $f \in W_n^1$. The basic reason for this is that functions in $W_p^1$, $p > n$, satisfy a Hölder condition whereas functions in $W_n^1$ can behave badly.

If $f$ is continuous on $I^n$, $f \in W_p^1(I^n)$, $p > n$, and $\|f\|$ is the $W_p^1$ norm of $f$ then $f$ satisfies a Hölder condition. Specifically, for every $x, y \in I^n$, $x \neq y$, we have

$$|f(x) - f(y)| \leq C(p,n) \cdot \|f\| \cdot |x - y|^{1 - \frac{n}{p}} .$$

From this inequality, it is not very difficult to see that

$$\Phi_i(f) \leq C(p,n)\|f\|, \quad i = 1,\ldots,n .$$

A computation, which is also not too difficult, yields the convergence in strong variation of the Bernstein polynomials $B_k^f$ to $f$. Consequently, $B_k^f$ converges to $f$ both in variation and in area.

The construction of a counter-example for $f \in W_n^1$ is much more subtle and interesting. Basic to the construction is the fact that, for $n > 1$, for every $\varepsilon > 0$ there is a non-negative $u \in W_n^1$ whose support is in the unit $n$ ball, such that $u(0) = 1$ and $\|u\| < \varepsilon$. This does not hold for $n = 1$. This implies that for every $\underline{a}$, $\delta > 0$, $m > 0$, $M > 0$ there is a $u \in W_n^1$ whose support is in the ball of center $\underline{a}$ and radius $\delta$ such that $u(a) = M$ and $\|u\| = m$.

Now

$$\Phi_1(B_k^f) = \int_{I^n} \left|\frac{\partial}{\partial x_1} B_k^f\right| dx$$

$$= \int_{I_n} \sum_{\nu_1=0}^{k-1} \sum_{\nu_2=0}^{k} \cdots \sum_{\nu_n=0}^{k} p_{\nu_1 k-1}(x_1) p_{\nu_2 k}(x_2) \cdots p_{\nu_n k}(x_n)$$

$$\cdot k \cdot \left\{f\left(\frac{\nu_1+1}{k}, \frac{\nu_2}{k}, \ldots, \frac{\nu_n}{k}\right) - f\left(\frac{\nu_1}{k}, \ldots, \frac{\nu_n}{k}\right)\right\} | dx$$

By putting these facts together in a judicious and delicate way, it is possible to obtain $u \in W_n^1(I^n)$ for which $\Phi_1(B_k^u)$ is an unbounded sequence. Accordingly $u$ is not the limit of $B_k^u$ in variation.

The Kantorovitch polynomials behave much better. Since these polynomials apply to functions in $L_1(I^n)$ we deal with functions in g ACT rather than in ACT. Indeed g ACT is simply $W_1^1$ without the continuity restriction. In other words, $f \in$ g ACT if there is an equivalent $g$ which is absolutely continuous in each variable for almost all values of the other variables and if the partial derivatives of $g$ are all summable. In contrast with Bernstein polynomials, for every $n$, $f \in L_1(I^n)$ implies $\Phi_i(P_k^f) \leq \Phi_i(f)$, $i = 1,\ldots,n$. It is also not hard to show that if $f \in$ g ACT on $I^n$ then $P_k^f$ converges strongly in variation to $f$, and hence converges in variation and in area. A related fact is that $P_k^f$ converges to $f$ in the $W_1^1(I^n)$ metric for $f \in$ g ACT $= W_1^1$.

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