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REMARKS ON UNIFYING PRINCIPLES IN REAL ANALYSIS

1. Unifying principles for proving fundamental theorems in real analysis

Several unifying principles for proving fundamental theorems in real analysis have been formulated in the mathematical literature. Such a principle is the principle of induction in continuum (cf. [7], [9], [10]). Some analogous principles are contained in [3], [4], [5], [8], and [11]. In this part of the paper we shall investigate the principles formulated in [8] and [11]. In particular the principle from [8] seems to be a very effective tool for simplifying proofs of some fundamental theorems in analysis. (See the second part of this paper.)

We shall formulate the principles from [8] and [11] for a chain (totally ordered set) $(X,<)$ having minimal element $(= a)$ and maximal element $(= b)$ and we shall suppose that $(X,<)$ has no gaps (i.e. for each two elements x, $y \in X$ with $x < y$ there exists a $z \in X$ such that $x < z$, $z < y$). In what follows we denote the interval topology on X by T.

The following properties (P_1) and (P_2) correspond to the principles from [8] and [11] respectively.

The chain $(X,<)$ is said to have the property (P_1) provided $<$ is the unique relation L on X satisfying the following conditions:

(A1) L is transitive, i.e. if $x \, L \, y$ and $y \, L \, z$, then $x \, L \, z$;

(A2) $L \subset <$;

(A3) L is locally valid, i.e. if $p \in X$, then there exists a neighborhood $V(p) \in T$ of the point p such that

$$x \in V(p),\ x < p \ \Rightarrow\ x \, L \, p \ , \quad \text{and}$$

$$x \in V(p),\ p < x \ \Rightarrow\ p \, L \, x \ .$$

The chain $(X,<)$ is said to have the property (P_2) provided each system S of closed intervals $[c,d] \subset X$ satisfying the conditions (B1) and (B2) contains $[a,b]$ $(= X)$, where

(B1) S is an additive system, i.e. if $[c,d] \in S$, $[e,f] \in S$ and $[c,d] \cap [e,f] \neq \emptyset$, then $[c,d] \cup [e,f] \in S$;

(B2) S is local, i.e. if $p \in X$, then there exists an interval $[c,d] \in S$ such that $[c,d]$ is a neighborhood of p (i.e. p belongs to $\mathrm{Int}[c,d]$).

We shall show in the Corollary after Theorem 1.1 that the properties (P_1) and (P_2) are equivalent and hence the principles of P. Shanahan and H. Leinfelder are equivalent.

The chain $(X,<)$ is said to be order-complete if each non-empty set $M \subset X$ with an upper bound has a supremum in X (cf. [6], p. 58). It follows from the proof of Theorem 2 in [8] that every order-complete chain without gaps has the property (P_1).

We shall show that each of the properties (P_1) and (P_2) is equivalent to the order-completeness of X.

Theorem 1.1. *Let* $(X,<)$ *be a chain without gaps, let* X *have minimal element* a *and maximal element* b.

(i) *The chain* X *has property* (P_1) *if and only if* X *is order-complete.*

(ii) *The chain* X *has property* (P_2) *if and only if* X *is order-complete.*

Corollary. *The chain* X *has property* (P_1) *if and only of it has the property* (P_2).

For the proof of Theorem 1.1 the following auxiliary result will be useful. (For the proof of the following Lemma 1.1 see [6], p. 58.)

Lemma 1.1. *Let* $(X,<)$ *be a chain without gaps. Then the topological space* (X,T) *is connected if and only if* $(X,<)$ *is order-complete*.

Proof of Theorem 1.1 (i) According to theorem 2 of [8] the relation $<$ is the unique relation on X satisfying the conditions (A1) - (A3) if and only if (X,T) is a connected space. Hence according to Lemma 1.1 the chain $(X,<)$ has property (P_1) if and only if it is order-complete.

(ii) If $(X,<)$ is order-complete, then X has property (P_2). This fact can be proved by the same procedure by which Lemma 1 in [11] is proved. We shall prove therefore only the fact that if X has property (P_2), then $(X,<)$ is order-complete.

It suffices to prove that if $(X,<)$ is not order-complete, then X does not have property (P_2). Let $H \subset [a,b]$, $H \neq \emptyset$. Suppose that H has no supremum in $X = [a,b]$. Denote by $B(H)$ the set of all upper bounds of the set H in X. Define the system S of closed intervals $[x,y] \subset X$ in the following way:

$$[x,y] \in S \iff [(x \in B(H)) \wedge (y \in B(H))]$$
$$\vee [(x \notin B(HH) \wedge (y \notin B(H))]$$

We shall show that S satisfies conditions (B1) and (B2).

Let $[x,y] \in X$, $[u,v] \in S$, $[x,y] \cap [u,v] \neq \emptyset$. Let for example $x < u < y < v$. (In the other cases we proceed in an analogous way.) If x, $y \in B(H)$, then $u \in B(H)$ and therefore also $v \in B(H)$, since $[u,v] \in S$. But then we have $[x,v] = [x,y] \cup [u,v] \in S$. If $x \notin B(H)$, $y \notin B(H)$, then $u \notin B(H)$ (since $u < y$) and therefore $v \notin B(H)$. We have again $[x,v] = [x,y] \cup [u,v] \in S$. Hence S satisfies (B1).

Let $p \in X$. If $p \notin B(H)$, then there exists an $x \in H$ such that $p < x$. But the interval $[a,x]$ is a neighborhood of the point p and $[a,x] \in S$. Let $p \in X$. Let $p \in B(H)$. Since H has no supremum, there exists an element $q \in X$, $q < p$, such that q is an upper bound of H $(q \in B(H))$. But then $[q,b] \in S$ and $[q,b]$ is a neighborhood of p. Hence S satisfies also the condition (B2). Since $a \notin B(H)$ and $b \in B(H)$, we see that $[a,b] \notin S$. Hence X does not have property (P_2). The proof is finished.

2. Two applications of the principle of H. Leinfelder

Using Theorem 1 from [8] we can give a simple proof of a known result in the theory of monotone functions. (See Theorem 2.1.) Recall that a function $f:(a,b) \to R$ is said to be increasing at the point $p \in (a,b)$ if there exists an open interval $I \subset (a,b)$ such that $x, p \in I$, $x > p$, implies $f(x) > f(p)$.

Theorem 2.1. <u>If the function</u> $f:(a,b) \to R$ <u>is increasing at each point</u> $p \in (a,b)$, <u>then it is increasing on the interval</u> (a,b).

<u>Proof</u>. On (a,b) define the relation L in the following way:

$$x \, L \, y \iff (x < y) \wedge (f(x) < f(y)) \, .$$

Then L satisfies conditions (A1) and (A2). We shall show that it also satisfies (A3). Let $c \in (a,b)$. Since f is increasing at c, there exists an interval $V(c) \subset (a,b)$ containing c such that

$$x \in V(c), \quad x < c \implies f(x) < f(c) \qquad (\text{i.e. } x \, L \, c) \, ,$$

$$x \in V(c), \quad c < x \implies f(x) < f(x) \qquad (\text{i.e. } c \, L \, x) \, .$$

But this shows that L satisfies (A3) too. Therefore according to Theorem 1 of [8] we have $L = <$ and so if $x, y \in (a,b)$, $x < y$, then $f(x) < f(y)$. The proof is finished.

We shall give another application of Theorem 1 of [8] in the theory of Lipschitzian functions. At first we shall introduce the definition of the concept of locally M-Lipschitzian functions. This definition is suggested by [2] and [1].

Definition 2.1. <u>Let</u> $I \subset R$ <u>be an interval and let</u> $M > 0$. <u>The function</u> $f:I \to R$ <u>is said to be</u> M-<u>Lipschitzian at the point</u> $p \in I$ <u>provided that there is a neighborhood</u> $V(p) \subset I$ <u>of the point</u> p <u>such that for each</u> $x \in V(p)$ <u>we have</u> $|f(x) - f(p)| < M|x - p|$.

Let us agree that $Lip_M \, 1$ stands for the class of all functions $f:I \to R$ that belong to the class Lip 1 with the constant M. Hence $f \in Lip_M \, 1$ if for each two points $x, y \in I$ we have $|f(x) - f(y)| < M|x - y|$.

Theorem 2.2. <u>Let</u> $f:I \to R$ <u>be</u> M-<u>Lipschitzian at each point</u> $p \in I$. <u>Then</u> $f \in Lip_M \, 1$.

<u>Proof</u>. Define the relation L_M on I in the following way:

$$x \, L_M \, y \iff (x < y) \wedge \left(-M \leqq \frac{f(x) - f(y)}{x - y} \leqq M\right)$$

Clearly L_M satisfies the conditions (A2) and (A3) of Theorem 1 of [8]. We shall show that it satisfies condition (A1) too. Let $x \, L_M \, y$, $y \, L_M \, z$. Then $x < y$ and $y < z$. Hence $x < z$. Further we have

$$(1) \qquad -M \leqq \frac{f(x) - f(y)}{x - y} \leqq M$$

and

$$(2) \qquad -M \leqq \frac{f(y) - f(z)}{y - z} \leqq M \, .$$

If $a/b < c/d$, then we have $a/b < a+c/b+d < c/d$. Therefore it follows from (1) and (2) that

$$-M \leqq \frac{f(x) - f(z)}{x - z} \leqq M \, .$$

Hence $x \, L_M \, z$.

According to Theorem 1 of [8] for each two points $x, y \in I$, $x < y$ we have $|f(x) - f(y)| < M|x - y|$. The proof is finished

Remark 2.1: Theorem 2.2 cannot be extended in the following way: "Let $f: I \to R$ be locally Lipschitzian at each point $p \in I$. (See [2], i.e. for each $p \in I$ there exists such an $M(p) > 0$ and a neighborhood $V(p) \subset I$ that for each $x \in V(p)$ we have $|f(x) - f(p)| < M(p)|x - p|$.) Then $f \in \mathrm{Lip}\,1$ where $\mathrm{Lip}\,1 = \cup_{M=1}^{\infty} \mathrm{Lip}_M\,1$". This fact follows from the following example.

Example 2.1. Put $I = [0,1]$, $f(0) = 0$ and $f(x) = x \sin 1/x$ for $x \in (0,1]$. Then f is evidently locally Lipschitzian at 0 and it has a finite derivative at each point $x \in (0,1]$. But it is easy to see that $f \notin \mathrm{Lip}\,1$. Put

$$x_n = (2\pi n + \frac{3\pi}{2})^{-1}, \quad y_n = (2\pi n + \frac{\pi}{2})^{-1} \quad (n = 1,2,\cdots) \, .$$

Then we have $|f(x_n) - f(y_n)| = x_n + y_n$ and it is evident that for a fixed $M > 0$ the inequality $x_n + y_n < M|x_n - y_n|$ cannot hold for each $n = 1,2,\cdots$.

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