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SOME PROBLEMS IN DIFFERENTIATION THEORY

Let (X,ρ_X) and (Y,ρ_Y) be complete metric space and let μ be a measure defined on a σ-algebra M which contains all Borel sets in X. We assume that there exists a differentiation base $(F,\Rightarrow)$ in X, where F is a family of open sets of finite, positive μ measure and a contraction $\Rightarrow$ of sequences of sets in F to points $x \in X$ is such that

(1) $I_n \Rightarrow x$ iff $x \in I_n$ for $n = 1,2,...$ and $\lim_{n\to\infty} d(I_n) = 0$, where $d(I_n)$ denotes the diameter of the set I_n ; and

(2) if $x \in X$, then there exists at least one sequence of sets of F which tends to x.

For a fixed set $E \in M$ and a point $x_0 \in X$ the upper (resp. lower) density of E at x_0 is

$$d^-(E,x_0) = \limsup_{I \Rightarrow x_0} \mu(E \cap I) / \mu(I)$$

$$\left(d_-(E,x_0) = \liminf_{I \Rightarrow x_0} \mu(E \cap I) / \mu(I) \right).$$

Here the notation $I \Rightarrow x_0$ is used to signify that we consider all possible sequences of open sets of F tending to x.

Definition 1. Let $f:X \to Y$ be a μ-measurable function. Then f satisfies the locally preponderantly Lipschitz condition at a point $x_0 \in X$, iff there exist a set $E \in M$, a number $\delta = \delta(x_0) > 0$ and a constant $L = L(x_0) > 0$ such that

$$\mu(E \cap I) / \mu(I) > 1/2 \text{ for all sets } I \in F \text{ containing } x_0$$

with $d(I) < \delta$ and

$$\rho_Y(f(x),f(x_0)) \leq L\, \rho_X(x,x_0) \text{ for every } x \in E .$$

Definition 2. ([1] and [2]). A function $f:X \to Y$ is [CG] if and only if for every closed set $C \subset X$ $(C \neq \Phi)$ there is an open set $U \subset X$ with $C \cap U \neq \Phi$ such that $f|_C$ is continuous on $C \cap U$.

Lemma 1. Let $A \subset X$ be a set and $f: C\ell\ A \to Y$ be a function ($A \neq \phi$ and $C\ell\ A$ denotes the closure of the set A). If the function f is not continous at a point $x \in C\ell\ A$, then there exist a number $c > 0$, a point $y \in C\ell\ A$ and a sequence of points $x_m \in A$ $(m = 1,2,\dots)$ which tends to y and such that $\rho_Y(f(y), f(x_m)) > c$ for every m.

Proof. Since f is not continuous at a point x, there exist a number $c > 0$ and a sequence of points $u_m \in C\ell\ A$ $(m = 1,2,\dots)$ which tends to x and such that $\rho_Y(f(x), f(u_m)) > 2c$ for every m. If $\lim_{t \to u_m, t \in A} f(t) = f(u_m)$ for every m, then there exists a sequence of points $x_m \in A$ such that $\rho_X(x_m, u_m) < 1/m$ and $\rho_Y(f(u_m), f(x_m)) < c$ for $m = 1,2,\dots$. Thus $\rho_Y(f(x_m), f(x)) \geq \rho_Y(f(u_m), f(x)) - \rho_Y(f(u_m), f(x_m)) > 2c - c = c$ for $m = 1,2,\dots$. If there exists an m such that either the limit $\lim_{t \to u_m, t \in A} f(t)$ does not exist or differs from $f(u_m)$, then Lemma 1 is fullfilled.

Theorem 1. If $f: X \to Y$ satisfies the locally preponderantly Lipschitz condition at every point $x \in X$, then f is [CG].

Proof. Let $C \subset X$ be a nonempty perfect set. For every natural number n let A_n be the set of all points $x \in C$ so that there is a set $E(x) \in M$ such that $\mu(E(x) \cap I)/\mu(I) > 1/2$ for every set $I \in F$ with $d(I) < 1/n$ and $\rho_Y(F(u), f(x)) \leq n\rho_X(x,u)$ for $u \in E(x)$. Since $C = \cup_n C\ell\ A_n$, by the Baire Category Theorem it suffices to show that the function $f|_{C\ell\ A_n}$ is continuous for each n. Suppose that for some n the function $f|_{C\ell\ A_n}$ is not continuous at a point $x \in C\ell\ A_n$. Then by Lemma 1 there exist a number $c > 0$, a point $y \in C\ell\ A_n$ and a sequence of points $x_m \in A_n$ which tends to y and such that $\rho_Y(f(y), f(x_m)) > c$ for every m. Let $\delta = \min(1/4n, c/4n, c/4L(y))$ and $I \in F$ be an open set such that $d(I) < \delta$ and $x \in I$ and $\mu(E(x) \cap I)/\mu(I) > 1/2$. There exists an m_o such that $x_{m_o} \in I$. Since $\mu(E(x_{m_o}) \cap I)/\mu(I) > 1/2$, there exists a point $u \in E(x_{m_o}) \cap E(y) \cap I$. Then

$$\rho_Y(f(y), f(x_{m_o})) \leq \rho_Y(f(y), f(u)) + \rho_Y(f(x_{m_o}), f(u)) \leq$$

$$L(y)\rho_X(x,u) + L(x_{m_o})\rho_X(x_{m_o},u) \leq L(y)\rho + n\rho \leq$$

$$L(y)c/4L(y) + nc/4n = c/2 < c .$$

This contradicts the fact that $\rho_Y(f(y), f(x_{m_o})) > c$.

II. Let X be an open, nonempty subset of k-dimensional Euclidean space R^k, let μ be Lebesgue measure in R^k, let $(\mathcal{F},\Rightarrow)$ be the ordinary differentiation basis ([3]) and let Y be a separable, Banach space.

Definition 3. A function $f:X \to Y$ is approximately differentiable at a point x_0 if there exist a set $E \in M$ containing x_0 with $d\_(E,x_0) = 1$ and a continuous linear operator $A:R^k \to Y$ $(A \in L(R^k,Y))$ such that

$$\lim_{h\to 0,\, x_0+h\in E} (f(x_0 + h) - f(x_0) - Ah)/|h| = 0 .$$

We shall write

$$f(x_0 + h) = f(x_0) + Ah + \epsilon_{x_0}(h)|h| \quad \text{for every} \quad h \in R^k$$

such that $x_0 + h \in E$ where $\lim_{h\to 0} \epsilon_{x_0}(h) = \epsilon_{x_0}(0) = 0$. Then the operator A is called the approximate derivative $f'_{ap}(x_0)$ of the function f at the point x_0.

Theorem 2. If a function $f:X \to Y$ is approximately differentiable at every point $x \in X$, then the approximate derivative $x \to f'_{ap}(x)$ is of Baire class 1.

Let us begin the proof with lemmas:

Lemma 2. If a function $f:X \to Y$ is approximately differentiable at every point $x \in X$ and f'_{ap} is not Baire class 1, then there there exist a perfect set $P \subset X$ $(P \neq \phi)$, an operator $A \in L(R^k,Y)$ and two numbers $s > r > 0$ such that $f|_P$ is continuous, the set $Q = \{x \in P: \|f'_{ap}(x) - A\|_L < r\}$ is of category 2 on every set $U \cap P$, where U is an open set and $U \cap P \neq \phi$ and the set $S = \{x \in P: \|f'_{ap}(x) - A\|_L \geq s\}$ is dense in P.

Proof. If $f'_{ap}:X \to L(R^k,Y)$ is not Baire class 1, then there exist a perfect set $P_1 \subset X$ $(P_1 \neq \phi)$ such that $f|_{P_1}$ is discontinuous at every point $x \in P_1$. Since the space $L(R^k,Y)$ is separable, there exists a set $\mathcal{A}$ = $\{A_1,A_2,\dots\}$ of operator $A_i \in L(R^k,Y)$ dense in $L(R^k,Y)$. For each point $x \in P_1$ there exist an operator $A(x) \in \mathcal{A}$ and two rational numbers $s(x) >$

$r(x) > 0$ such that $\|f'_{ap}(x) - A(x)\|_L < r(x)$ and $x \in Cl\ \{t \in P_1: \|f'_{ap}(t) - A(x)\|_L \geq s(x)\}$. Let s and t be positive, rational numbers and n be a positive integer such that the set $P_2 = \{x \in P_1: A(x) = A_n = A,\ r(x) = r,\ s(x) = s\}$ is of category 2 in P_1. The set $P_3 = \{x \in Cl\ P_2: P_2$ is of category 2 (relative to P_1) at point $x\}$ is perfect. Since f is approximately differentiable, by Theorem 1 f is [CG] and there exists an open set $U \subset X$ with $P_3 \cap U \neq 0$ such that $f|P_3$ is continuous on $P_3 \cap U$. Let V be an open set such that $Cl\ V \subset U$ and $V \cap P_3 \neq \phi$. The set $P = Cl\ V \cap P_3$ is perfect, $f|_P$ is continuous, the set $Q = \{x \in P: \|f'_{ap}(x) - A\|_L < r\}$ is of category 2 on every set $W \cap P$ where W is an open set and $W \cap P \neq \phi$ and the set $S = \{x \in P: \|f'_{ap}(x) - A\|_L \geq s\}$ is dense in P. This completes the proof.

Lemma 3. If $\|f'_{ap}(x)\|_L \geq s$ for every $x \in S \neq \phi$ and if s_1 is a number such that $0 < s_1 < s$, then there exists a positive number q such that $d_-(\{h \in R^k: h \neq 0, \|f'_{ap}(x)h/|h|\| > s_1\}, 0) \geq q$ for all $x \in S$.

Proof. Let x be a point of S and let s_2 be such that $s_1 < s_2 < s$. Since $\|f'_{ap}(x)\|_L \geq s$, there exists $h_o \in R^k$ such that $\|h_o\| = 1$ and $\|f'_{ap}(x)h_o\| > \|f'_{ap}(x)\|_L - (s_2 - s_1)/2$. If $h \in R^k$, $\|h\| = 1$ and $\|h - h_o\| < 1 - s_2/s$, then $\|f'_{ap}(x)h\| \geq \|f'_{ap}(x)h_o\| - \|f'_{ap}(x)h - f'_{ap}(x)h_o\| \geq \|f'_{ap}(x)h_o\| - \|f'_{ap}(x)\|_L\ |h - h_o| > \|f'_{ap}(x)\|_L\ (1 - |h - h_o|) + - (s_2 - s_1)/2 > s\ s_2/s - (s_2 - s_1)/2 > s_1$.

Let $q = d_-(\{h \in R^k: h \neq 0$ and $|h/|h| - h_o| < 1 - s_2/s\}, 0)$. Then $q > 0$ and $d_-(\{h \in R^k: h \neq 0,\ \|f'_{ap}(x)h/|h|\| > s\}, 0) \geq d_-(\{h \in R^k: h \neq 0$ and $|h/|h| - h_o| < 1 - s_2/s\}, 0) \geq q$. This completes the proof.

Lemma 4. Let $f: X \to Y$ be a function and $A \subset X$ be a set such that $f|_A$ is continuous at a point $s \in A$. If

(1x) $\mu(\{u \in I: \|(f(u) - f(x))/|u-x| - a\| < \epsilon\}/\mu(I) > \delta$

where $a \in Y$, $I \in F$, $x \in I$ and δ, ϵ are some positive constants, then there exists a nonempty open set $U \subset X$ such that $x \in U$ and the condition (1z) is satisfied for each $z \in U \cap A$.

Proof. For every point $t \in E = \{u \in I: \|(f(u) - f(x))/|u-x| - a\| > \epsilon\}$ there exists a rational number $r(t) > 0$ such that

$\|(f(t) - f(z))/|t-z| - a\| > \epsilon$ for all $z \in A \cap I$ with $|z-x| < r(t)$. Since $\mu(E) > \delta\mu(I)$, we obtain $\mu(\{t \in E: r(t) \geq r\}) > \delta\mu(I)$ for any $r > 0$. Then condition (1z) is satisfied for each $z \in I \cap A$ such that $\delta_X(z,x) < r$.

Proof of Theorem 2. If $\dot{f}_{ap}$ is not Baire class 1, then by Lemma 2 there exist a perfect set $P \subset X$ $(P \neq \Phi)$ and an operator $A \in L(R^k,Y)$ and numbers $s > r > 0$ such that $f|_P$ is continuous, the set $Q = \{x \in P: \|\dot{f}_{ap}(x) - A\|_L < r\}$ is of category 2 on every open set $U \cap F$ with $U \cap P \neq \Phi$ and the set $S = \{x \in P: \|\dot{f}_{ap}(x) - A\|_L \geq s\}$ is dense in P. We can assume that $A = 0$, since in other case we consider the function $f - A$. By Lemma 3 there exists a positive number q such that $d\_(\{h \in R^k: h \neq 0, \|\dot{f}_{ap}(x)h/|h|\| > (s+r)/2\},0) \geq q$ for every $x \in S$. Now for each $x \in Q$ there is a positive rational number $\delta(x)$ such that $\mu(\{t \in I: \|f(t) - f(x)\|/|t-x| < r\})/\mu(I) > 1 - q/2$ for every $I \in \mathcal{F}$ with $x \in I$ and $d(I) < \delta(x)$. Since the set Q is of category 2 in P, there exists a number $\delta > 0$ such that the set $T = \{x \in Q: \delta(x) = \delta\}$ is of category 2 in P. Consequently there exists an open set V such that $V \cap P \neq \Phi$ and $T \cap V$ is dense in $V \cap P$. Let $x \in S \cap V$ be a point and $I \in \mathcal{F}$ be a set such that $x \in I$, $d(I) < \delta$ and $\mu(\{t \in I: \|(f(t) - f(x))/|t-x|\| < r\})/\mu(I) > q/2$. By Lemma 4 there exists an open set $W \subset V$ such that $x \in W$ and the condition (1z) (or $a = 0$, $\epsilon = (s+r)/2$ and $\delta = q/2$) is satisfied for each $z \in W \cap P$. But T is dense in $V \cap P$, so there exists a point $y \in T \cap I \cap W$ and $\mu(\{t \in I: \|f(t) - f(y)\|/|t-y| < r\})/\mu(I) > 1 - q/2$, in contradiction with (1y). This completes the proof.

Remark. If $X = Y = R$, then Theorem 2 is proved in [4] by Tolstoff.

Definition 4. A function $f: X \to Y$ is preponderantly differentiable at a point $x \in X$ if there exist: a set $E(x) \in M$, a number $\delta = \delta(x) > 0$ and a linear operator $A: R^k \to Y$ such that $\mu(E(x) \cap I) > \mu(I)/2$ for all sets $I \in \mathcal{F}$ containing x with $d(I) < \delta$ and

$$\lim_{h \to 0,\, x+h \in E(x)} (f(x+h) - f(x) - Ah)/|h| = 0.$$

Then the operator A is called the preponderant derivative $f'_{pr}(x)$ of the function f at the point x.

Theorem 3. If $k = 1$ and if the function $f:X \to Y$ is preponderantly differentiable at every point $x \in X$, the preponderant derivative f'_{pr} is of the first class of Baire.

The proof of this theorem is similar to the proof of the Tolstoff Theorem 1 in [4].

Problem 1. Let $X \subset R^k$ $(k > 1)$ be an open nonempty set and $f:X \to Y$ be preponderantly differentiable at every point $x \in X$. Must f'_{pr} be Baire 1?

Problem 2. If $X \subset R^k$ $(k \geq 1)$ is an open nonempty set and if μ is a measure for which all bounded open nonempty sets have positive finite measure and if $f:X \to R$ is ordinarily approximately differentiable at every point $x \in X$, must the ordinary approximate derivative f'_{ap} be of Baire 1 class?

III. Let $f:[0,1] \times [0,1] \to R$ be a function such that all sections $f^y(t) = f(t,y)$ are increasing.

Theorem 4. ([5]) If all sections $f_x(t) = f(x,t)$ are almost everywhere continuous (a.e. differentiable) [pointwise discontinuous], then the function f is a.e. continuous (a.e. differentiable in Frechet sense) [pointwise discontinuous].

Theorem 5. ([5]) If all sections f_x and f^y are increasing, then the set $D(f)$ of all discontinuity points of f is such that the sets $D_1(f) = \{x:(D(f))_x$ is not enumerable$\}$ and $D_2(f) = \{y: (D(f))^y$ is not enumerable$\}$ are at most enumerable.

Some characterizations of the sets $D(f)$ are known if all f_x and f^y are increasing ([6]).

Problem 3. If all sections f_x are a.e. continuous (a.e. differentiable) [pointwise discontinuous] and if all sections f^y are monotone (increasing or decreasing), must the function f be a.e. continuous (a.e. differentiable in Frechet sense) [pointwise discontinuous]?

Problem 4. Moreover, if all sections f_x and f^y are monotone, must the sets $D_1(f)$ and $D_2(f)$ be at most enumerable?

Problem 5. What is a necessary and sufficient condition for an F_σ and first category set to be the set $D(f)$ of all discontinuity points of a function f such that all sections f_x and f^y are monotone?

Problem 6. What is a necessary and sufficient condition for a set $E \subset [0,1]^2$ to be the set of all differentiability points of a function $f\colon [0,1]^2 \to R$ such that all sections f_x and f^y are increasing (monotone)?

IV. The functions $f(x) = x$ and $g(x) = -x$ are differentiable, but the function $h = \max(f,g)$ is not differentiable at 0. Thus the family of all differentiable functions is not a lattice of functions.

Theorem 6. If functions $f,g\colon R \to R$ are differentiable on an open interval and if $h = \max(f,g)(\min(f,g))$ is not differentiable at a point $x \in U$, then there exists a number $r > 0$ such that $(x-r,x+r) \subset U$ and the function h is differentiable at every point $u \in (x-r,x) \cup (x,x+r)$.

Proof. Obviously if the function h is not differentiable at a point $u \in U$, then $f(u) = g(u) = h(u)$. If for every $r > 0$ there exists a point $x_r \in (x-r,x) \cup (x,x+r)$ where h is not differentiable, then $\lim_{r\to 0} (h(x_r) - h(x))/(x_r - x) = \lim_{r\to 0} f(x_r) - f(x))/(x_r - x) = f'(x) = g'(x) = \lim_{r\to 0} (g(x_r) - g(x))/(x_r - x)$. This gives that $\lim_{u\to x} (h(u) - h(x))/(u - x) = f'(x) = g'(x) = h'(x)$, contrary to the choice of x. The contradiction completes the proof.

Corollary. If $h = \max(f,g)$ $(h = (\min(f,g))$ where the functions f and g are differentiable, then the set of all points where h is not differentiable is discrete.

Theorem 7. Let $E \subset R$ be a discrete set. Then there are differentiable functions $f,g:R \to R$ such that function $h = \max(f,g)$ is differentiable at every point $x \notin E$ and is not differentiable at every point $x \in E$.

Proof. Let $E = \{a_1,a_2,\ldots\}$, where $a_i \neq a_j$ if $i \neq j$. For every $n = 1,2,\ldots,$ there is an interval $(a_n - r_n, a_n + r_n)$ such that $[a_n - r_n, a_n + r_n] \cap [a_m - r_m, a_m + r_m] = \Phi$ if $n \neq m$. For any n let $h_n:R \to [0,2^{-n}]$ be a function such that

1) $h_n(x) = 0$ for $x \in (-\infty, a_n - r_n] \cup [a_n + r_n, \infty)$;

2) h_n is differentiable at every point $x \neq a_n$ and $|h_n'| \leq 2^{-n}$;

3) h_n is not differentiable at a_n ;

4) $h_n = \max(f_n,g_n)$, where the functions f_n and g_n are differentible and such that $f_n(x) = g_n(x) = 0$ for $x \notin [a_n - r_n, a_n + r_n]$ and $|f_n| \leq 2^{-n}$ and $|g_n| \leq 2^{-n}$ and $\max(|f_n'|,|g_n'|) \leq 2^{-n}$. Then the functions $f = \Sigma_n f_n$, $g = \Sigma_n g_n$ and $h = \max(f,g)$ satisfy the desired properties.

Problem 7. What is the smallest lattice of functions containing all differentiable functions? Is it the family of all continuous functions differentiable at every point except perhaps at the points of a set which is a finite union of discrete sets?

Problem 8. What is the smallest lattice of functions containing all derivatives (approximately derivatives) [preponderant derivatives] {Baire 1, Darboux functions} {monotone functions} {Riemann integrable derivatives}?

Problem 9. What is the smallest algebra of functions containing all almost everywhere continuous derivatives? Is it the family of all a.e. continuous Baire 1 functions?

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