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SHARPNESS OF SOME GRAPH CONDITIONED THEOREMS ON BOREL 1 SELECTORS

The purpose of the present note is to provide a negative answer to questions 2, 5 and 10 posed by J. Ceder and S. Levi in [3]. Indeed, this note can be viewed as a continuation of [13] where questions 4, 6 and 7 were answered.

First we give some preliminaries. A multifunction $F:X \to Y$ between topological spaces X and Y is a map from X into the nonempty subsets of X. We say that F is in lower (resp. upper) class α, if $F^{-}(U) = \{x \in X : F(x) \cap U \neq \phi\}$ is a Borel set of additive (resp. multiplicative) class α in X for every open (resp. closed) set in Y. For information concerning the above-mentioned classification see [9] and [6]. A goal of Borel 1 selector theory is to find weak hypotheses of F guaranteeing the existence of a Borel 1 selector f for F, i.e. an ordinary Borel 1 function f from X to Y satisfying $f(x) \in F(x)$ for all x.

In order to obtain a Borel 1 selector for F there are three sorts of hypotheses we might impose:

1) We might require each value $F(x) \subset Y$ of F to be a set of some special kind: closed, sigma-compact, G_δ, relatively nonmeager, etc.

2) We might require the graph of F.

$$\mathrm{Gr}\, F = \{(x,y) : y \in F(x)\} \subset X \times Y$$

to be a set of some special kind.

3) We might require F to belong to some lower or upper class.

Throughout this paper we will be imposing conditions on the topological nature of $\mathrm{Gr}\, F$. We begin with the following general result.

Theorem 1 (Debs [4]). Let $F:X \to Y$ where X is perfectly normal and Y is Polish. Suppose that:

a) F is in lower class $\alpha \geq 1$, and

b) there exist a countable family of open sets $\{U_{nk};\ k = 1,2,\ldots;\ n = 1,2\ldots\}$ and a countable family of ambiguous class α (i.e. simultaneously of additive class α and multiplicative class α) sets $\{A_{nk};\ k = 1,2,\ldots;\ n = 1,2,\ldots\}$ such that

$$\operatorname{Gr} F = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} (A_{nk} \times U_{nk}) .$$

Then F has a Borel α selector.

Observe that values of F in Theorem 1 are G_δ sets and that each multifunction F with metrizable domain X and with a G_δ graph fullfils Debs condition b) automatically. On the other hand if the values of F are closed in Y, then assumption b) in Theorem 1 can be omitted by virtue of the famous Fundamental Selection Theorem of K. Kuratowski and Cz. Ryll-Nardzewski. Note that though no graph hypothesis is made in the Fundamental Selection Theorem, it follows almost trivially from the other hypotheses that in our case $\operatorname{Gr} F$ is in fact of multiplicative class α in $X \times Y$. Thus the following question, posed in [3], arises:

Question 2 (original numeration). Let $F : X \to Y$ where X is metric and Y is Polish. If F is in lower class 1 and $\operatorname{Gr} F$ is an $F_{\sigma\delta}$ (or a $G_{\delta\sigma}$) set, does F have a Borel 1 selector?

A metrizable space X is called Souslin if it is a continuous image of some Polish (i.e. a second countable completely metrizable) space. If X is a Polish space, then $A \subset X$ is called cosouslin if $X - A$ is a Souslin set. We refer the reader unfamiliar with the theory of Souslin sets to K. Kuratowski, Topology I, Academic Press 1966 for facts useful in the proof of Example 2 mentioned below (e.g. continuous 1-1 images of Borel sets are Borel, graphs of Borel functions are Borel sets, etc.). Also Y. Moschovakis, Descriptive set theory, Amsterdam 1980, is an appropriate reference. Note that a negative answer to Question 2 in the $G_{\delta\sigma}$ case is implied by some results of Z. Grande [7]:

Example 1. Let R denote the real line. There exists a multifunction $F:R \to R$ in lower class 0 with $\mathrm{Gr}\, F \in G_{\delta\sigma}(R^2)$ and with open values which admits no Borel 1 selector.

Proof. Let $g:R \to R$ be a Borel 2 function such that $\mathrm{Gr}\, g$ intersects the graph of each Borel 1 functions F existing in compliance with [7]. Put $F(x) = R - \{g(x)\}$ and observe that F has the desired properties.

Without requiring that each value of F be relatively nonmeager one can find an example of a multifunction having no Borel measurable selector but satisfying all assumptions of Question 2. This counterexample also gives a negative answer to the following.

Question 10 [3]. Let $F:X \to Y$ where Y is Polish. Does there exist a Borel 2 selector when $\mathrm{Gr}\, F$ is an $F_{\sigma\delta}$ set?

Example 2. There exist Polish spaces X and Y and a lower semicontinuous (i.e. in lower class 0) multifunction $F:X \to Y$ having F_σ graph and F_σ values but no Borel selector.

Proof. We adopt some constructions from [8], [10] and [2]. Let $X = Y = N^N$ denote the set of all sequences of positive integers. Endowed with the product of discrete topologies on N, N^N becomes a homeomorph of the space of irrationals. Observe that

$$d(x,y) = \begin{cases} k^{-1} & \text{if } x_k \neq y_k \text{ and } x_i = y_i \text{ whenever } i < k \\ 0 & \text{if } x_i = y_i \text{ for all } i \end{cases}$$

defines a complete metric on Y and that $\{y \in Y : y_n = z_n$ for all but finitely many indices $n\}$ serves as a countable dense subset of Y whenever $z \in Y$.

In the first step by using an argument due to P. Novikov we construct a multifunction $G:X \to Y$ having closed graph but no Borel selection (cf. [2]). Let C_1 and C_2 be a pair of disjoint cosouslin subsets of X which are not Borel separable. (See [10].) Let $A_j = X - C_j : j = 1,2$. Observe that A_1 and A_2 are Souslin sets whose union is X. Let F_j be a closed subset of

$X \times Y$ which projects exactly onto $A_j : j = 1,2$. Define $G(x) = \{y \in Y : (x,y) \in F_1 \cup F_2\}$ and suppose that g is some Borel selector for G. Put $T = \{x \in X : (x,g(x)) \in F_2 - F_1\}$. Obviously T is a Borel subset of X as a continuous and bijective (on $Gr\ g$) image of $Gr\ g - F_1$. It is easily verified that $C_1 \subset T$ and $T \cap C_2 = \phi$ which contradicts the fact that C_1 and C_2 are not Borel separable. Thus G has no Borel selector.

Let $S = N^0 \cup N^1 \cup N^2 \cup N^3 \cup \ldots \cup N^k \cup \ldots$ the monoid of all finite sequences of positive integers with concatenation $*$ as a composition law and with the empty sequence $e \in N^0$ as a neutral element. This monoid acts transitively on Y according to the formula:

$$F(x) = \bigcup_{s \in S} [s*G(x)] \quad \text{where}$$

$s*G(x) = \{s*y : y \in G(x)\}$. Since $Gr[s*G]$ as well as $Gr\ G$ is closed in $X \times Y$, $Gr\ F \in F_\sigma(X \times Y)$. The values of F are dense F_σ sets. In fact if $y \in F(x)$, then $\{s*y : s \in S\} \subset F(x) \subset Y$. Thus it is easy to check that F is lower semicontinuous. Now assume by way of contradiction that there is a Borel selector f for the above defined multifunction F.

Define:

$$A_e = \{x \in X : f(x) \in G(x)\}, \text{ and for each } s \in S \text{ by reccurence}$$

$$A_s = \{x \in X : f(x) \in s*G(x)\} - \bigcup_{p \leqslant s} A_p .$$

The sign $\leqslant$ means here the usual lexicographic order on S. Observe that $\{A_s : s \in S\}$ is a countable family of pairwise disjoint Borel subsets of X and that $\cup_{s \in S} A_s = X$.

Define:

$$g(x) = \begin{cases} f(x) = (f_1(x), f_2(x), \ldots, f_nF(x) \ldots) & \text{if } x \in A_e \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ (y_1, y_2, \ldots, y_n, \ldots) = (f_{k+1}(x), f_{k+2}(x), \ldots, & \\ \quad f_{k+n}(x), \ldots) & \text{if } x \in A_s \text{ where } s = (s_1, s_2, \ldots, s_k). \end{cases}$$

Observe that $g(x) \in G(x)$ for all $x \in X$. Since all restrictions $g|A_s$ are Borel measurable, g would be a Borel selector of G which contradicts the fact stated in the first step that G has no Borel selector.

In **Example** 9 in [3] an open-valued multifunction in upper class 1 with G_δ graph but without Borel 1 selector is constructed. Our next example shows that assumption a) in Theorem 1 cannot be weakened even if F is compact-valued.

Example 3. There exists a compact-valued multifunction $F:R \to R$ belonging to upper class 1 which has a G_δ graph but no Borel 1 selector.

Proof. Fix two disjoint countable dense subsets $D_1, D_2 \subset R$ and define

$$F(x) = \begin{cases} +1 & \text{if } x \in D_1 \\ -1 & \text{if } x \in D_2 \\ \{-1,+1\} & \text{if } x \in R - (D_1 \cup D_2). \end{cases}$$

Observe that $\text{Gr } F = (R-D_1) \times \{-1\} \cup (R-D_2) \times \{1\} \in G_\delta(R^2)$ and that $F^{-}(K) \in \{\phi, R-D_1, R-D_2, R\} \subset G_\delta(R)$ whenever K is closed. Let f be any selector for F. Observe that f must be totally discontinuous and therefore not a Borel 1 function.

The multifunction G from Example 2 shows that the existence of Borel 1 selectors is not implied by closed $\text{Gr } G$. For multifunctions with values in sigma-compact spaces the situation is quite different.

Theorem 2 ([3]; Th. 4). Let $F:X \to Y$ where X and Y are metric spaces. If Y is sigma compact and $\text{Gr } F \in F_\sigma(X \times Y)$, then F is in lower class 1 and F has a Borel 1 selector.

Remark. The proof of Theorem 2 remains valid if X is assumed to be perfectly normal only and Y is a countable union of metrizable compact sets or equivalently Y is a continuous image of some closed subset of R. In particular Y may be any separable, metrizable, locally compact space as well as the weak dual of some separable, metrizable, locally convex, linear space. Theorem 2 is stated in [3] together with the following question:

Question 5. Let $F:X \to Y$ where X and Y are metric spaces. If each $F(x)$ is sigma compact and $\mathrm{Gr}\, F$ is an F_σ, does there exist a Borel 1 selector for F?

The answer is negative even in the case where $\mathrm{Gr}\, F$ is closed and values $F(x)$ are compact:

Example 4. There exist metric spaces X and Y and a multifunction F from X to Y with closed graph and compact values having no L-measurable selector.

Proof. Let X denote the real line with the Euclidean topology T_X and Y the real line with the discrete topology T_Y. Both T_X and T_Y are metrizable. Put $F(x) = \{x\}$ and observe that $\mathrm{Gr}\, F$ is $T_X \otimes T_X$ - closed and thus it is also $T_X \otimes T_Y$ - closed since $T_X \otimes T_X \subset T_X \otimes T_Y$. Let $Z \subset X$ be a nonmeasurable subset. Obviously Z is T_Y - open and $f^{-1}(Z) = Z$ is nonmeasurable where $f(x) = x$ is the sole selector for F.

It should be noted that if X is a Souslin space (i.e., a continuous image of N^N from Example 2) and if Y is a Polish space, then the multifunction in Question 5 admits a Borel α selector for an unspecified α. Indeed we have the following deep theorem.

Theorem 3 ([12]). Let X be a Souslin space and Y a Polish space. If $F:X \to Y$ has a Borel graph $\mathrm{Gr}\, F$ and if $F(x)$ is sigma compact for every $x \in X$, then F has a Borel measurable selector. Moreover there exists a sequence $B_1, B_2, \ldots$ of Borel sets in $X \times Y$ such that

$$\mathrm{Gr}\, F = \bigcup_{n=1}^{\infty} B_n$$

and $B_n(x) = \{y \in Y : (x,y) \in B_n\}$ is compact for all $n \in N$ and $x \in X$. (See also [14], Th. 2.3.)

Note that the complexity of a selector f in the framework of Theorem 3 cannot be estimated by and $\alpha < \Omega$ as the following example shows.

Example 5. Let $\alpha < \Omega$ be an arbitrary ordinal number. Then there exist Polish spaces X and Y and a compact-valued multifunction $F:X \to Y$ with closed graph but having no Borel α selector.

The following theorem is to be invoked for $\alpha + 1$ in the proof of our Example 5.

Theorem 4 ([11], Th. 2.3. See also [1] for simple proof.) Let X be a Polish space, $B = \{U_1, U_2, \ldots\}$ a countable basis for X, $A = \{A_1, A_2, \ldots\}$ a countable family of ambiguous α sets closed under complementation $\alpha \geq 1$. Then there is a countable family $C = \{C_1, C_2, \ldots\}$ of ambiguous β sets, $\beta < \alpha$, such that the topology T_α generated by $B \cup A \cup C$ on X is Polish.

Proof of Example 5. Let $X = Y = N^N$ be the Baire space from Example 2. Fix some subset $Z \subset X$ of Borel additive class α not belonging to the multiplicative class α. Such a set exists by virtue of [5]. Then put $A = \{Z, X\text{-}Z\}$. By Theorem 4 we get the new Polish topology $T_{\alpha+1} = T_Y$ such that T_Y is finer than the original topology T_X and every T_Y - open set is an additive class $\alpha + 1$ set with respect to the original topology T_X. Define $F:X \to Y$ by the formula $F(x) = \{x\} = \{f(x)\}$ and observe that $\text{Gr } F$ is T_X T_Y - closed and that $f^{-1}(Z)$ is not in the multiplicative T_X - class α while Z is T_Y - closed. Since $f:x \to x$ is the sole selector for F, F has the desired properties.

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