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WHITNEY SETS AND SETS OF CONSTANCY ON A PROBLEM OF WHITNEY

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1. Let H be a connected subset of $\mathbb{R}^n$. H is said to be a Whitney set (W-set) if there exists a non-constant function $f:H \to \mathbb{R}$ such that

$$(1) \qquad \lim_{\substack{x \to x_0 \\ x \in H}} \frac{|f(x)-f(x_0)|}{|x-x_0|} = 0$$

holds for every $x_0 \in H$.

On the other hand, H is said to be a set of constancy (or a C-set) if it is connected and not a W-set.

It is easy to see that any rectifiable continuous curve is a C-set. However, there are simple arcs which are W-sets as was first shown by Whitney [5]. Later, several other examples where found for W-arcs ([1], [3]). In his paper [5] Whitney raised the following problem: how far need a simple arc be from rectifiability in order to be a W-set.

As it turns out, rectifiability is not the proper approach to find a characterization of W-sets. It was proved by Choquet [3] that, the graph of any continuous function

$f:[a,b] \to \mathbb{R}$ is a C-set. According to a result of Besicovitch and Schoenberg [1], the Hausdorff dimension of a graph can be 2, showing that even this very strong non-rectifiability does not imply the W-property.

In this paper we construct a simple example of a W-arc γ, where the non-constant function with identically zero derivative is the inverse of the parametrization of γ. We also provide a sufficient condition for the W-property to hold and then, generalizing Choquet's result, a sufficient condition for the C-property to hold. The exact characterization of W-sets or C-sets remains open.

2. Theorem 1. There exists a continuous one-to-one mapping $\varphi:[0,1] \to \mathbb{R}^2$ such that

$$\lim_{t \to t_0} \frac{|\varphi(t)-\varphi(t_0)|}{|t-t_0|} = \infty \tag{2}$$

holds (uniformly) for every $t_0 \in [0,1]$.

Remark. This theorem automatically proves that $\varphi([0,1]) = \gamma$ is a W-arc, since $f = \varphi^{-1}$ is a non-constant function on γ with (1).

Proof. For any given square Q, the shaded areas on fig. 1. are called configurations A and B applied in Q, respectively.

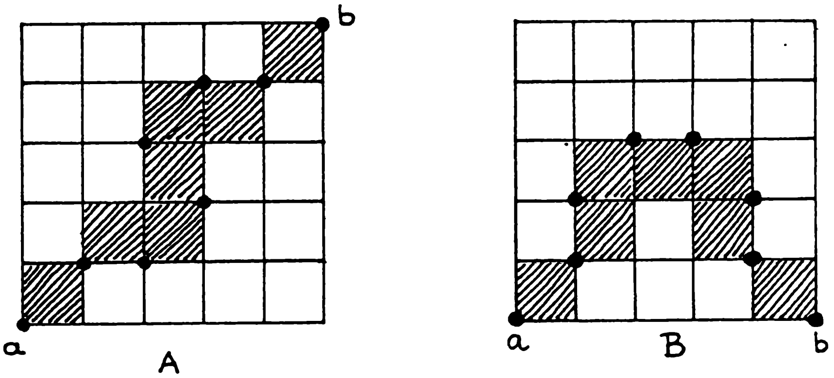

figure 1.

We are going to define a sequence of chains

$$C_n = \{Q_o^n, \ldots, Q_{7^n-1}^n\}$$

of non-overlapping squares Q_j^n such that $Q_{i-1}^n \cap Q_i^n$ is either a common vertex or a common side of Q_{i-1}^n and Q_i^n $(i=1,2,\ldots,7^n-1)$. We start with the unit square Q_o^o and put

$$C_o = \{Q_o^o\}.$$

Suppose that the n^{th} chain C_n has been defined. We select two different vertices a_i and b_i of each Q_i^n $(i=0,\ldots,7^n-1)$ such that $b_i = a_{i+1}$ for every $i=0,1,\ldots,7^n-2$. (A possible selection for C_1 is shown on figure 1.) In each of the squares Q_i^n we connect the vertices a_i and b_i by configuration A or B applied in Q_i^n and denote the squares of this configuration in the natural order Q_{7i+j}^{n+1} $(j=0,\ldots,6)$. We put

$$C_{n+1} = \{Q_i^{n+1};\ i=0,\dots,7^{n+1}-1\}.$$

In this way we define Q_i^n for every $n \in \mathbb{N}$ and $0 \le i < 7^n$. It is easy to check that, for every n,

(i) $Q_i^n \cap Q_{i+1}^n \neq \emptyset \quad (i=0,\dots,7^n-2)$,

and

(ii) $\operatorname{dist}(Q_i^n, Q_j^n) \ge 5^{-n} \quad (0 \le i,j \le 7^n-1,\ |i-j| \ge 3)$.

(As for (ii), $|i-j| \ge 3$ implies $Q_i^n \cap Q_j^n = \emptyset$ and hence these squares are separated by a strip of width 5^{-n}.)

Now we define the map φ as follows. For every $t \in [0,1]$ and $n \in \mathbb{N}$ we choose $i_n = i_n(t) \in \mathbb{N}$ such that

$$\frac{i_n}{7^n} \le t \le \frac{i_n+1}{7^n}$$

and define $\varphi(t) = \bigcap_{n=0}^{\infty} Q_{i_n}^n$. It is obvious that φ is well-defined, one-to-one and continuous on $[0,1]$. Let $0 \le t_1 < t_2 \le 1$ be fixed. There is $n \ge 1$ such that

$$\frac{3}{7^n} < t_2 - t_1 \le \frac{3}{7^{n-1}} .$$

Consider the squares Q_i^n and Q_j^n of the n^{th} chain C_n covering the points $\varphi(t_1)$ and $\varphi(t_2)$, respectively. Now, $\frac{3}{7^n} < t_2 - t_1$ implies $|i-j| \ge 3$ and hence by (ii) we have $|\varphi(t_1) - \varphi(t_2)| \ge 5^{-n}$. Thus

$$\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_2-t_1|} \ge \frac{5^{-n}}{3\cdot 7^{-n+1}} = \frac{1}{21}\cdot\left(\frac{7}{5}\right)^n,$$

and the proof is complete.

Our next result generalizes Theorem 1. Let $\varphi:[0,1] \to \mathbb{R}^n$ be an arbitrary mapping and put

$$H_\varphi = \{t \in [0,1]: \lim_{s\to t} \frac{|\varphi(s)-\varphi(t)|}{|s-t|} = \infty\} .$$

In Theorem 1 we had $H_\varphi = [0,1]$.

Theorem 2. Let φ be the parametrization of a simple arc $\gamma \subset \mathbb{R}^n$ and suppose $\lambda(H_\varphi) > 0$. Then γ is a W-set.

Proof. H_φ is obviously a measurable set, since φ is continuous. Let P denote a nowhere dense perfect set such that $P \subset H_\varphi$ and $\lambda(P) > 0$. If q is a density point of P then by a well known lemma of Zahorski ([6] or [2], p.28) there exists an approximately continuous function g such that $0 \le g \le 1$, $g(q) = 1$, and $g(x) = 0$ $(x \notin P)$. Let $G = \int g$, then G is differentiable and constant on the intervals contiguous to P , but not identically constant, since $G'(q) = 1 \neq 0$. Now we define

$$f(z) = G(\varphi^{-1}(z)) \qquad (z \in \gamma).$$

Let $z_0 \in \gamma$ be fixed and denote $z = \varphi(t)$, $z_0 = \varphi(t_0)$. Then

$$\frac{|f(z)-f(z_0)|}{|z-z_0|} = \left|\frac{G(t)-G(t_0)}{t-t_0}\right| \cdot \frac{|t-t_0|}{|\varphi(t)-\varphi(t_0)|} .$$

If $t_0 \in P$ then $t_0 \in H_\varphi$ and hence

$$\frac{|t-t_o|}{|\varphi(t)-\varphi(t_o)|} \to 0 \qquad (t \to t_o) ,$$

thus $\lim_{z\to z_o} \frac{|f(z)-f(z_o)|}{z-z_o} = 0$.

If $t_o \notin P$ then G is constant in a neighbourhood of t_o , thus $f(z)-f(z_o) = 0$ in a neighbourhood of z_o and again

$$\lim_{z\to z_o} \frac{|f(z)-f(z_o)|}{|z-z_o|} = 0 .$$

Finally, f is not constant, since G is not constant.

3. In this section we give a sufficient condition for the C-property. This generalizes a theorem of Choquet [3] stating that the graph of any continuous function is a C-curve.

<u>Theorem 3.</u> Let $\varphi:[a,b] \to \mathbb{R}^n$ be continuous and let

$$E = \{\varphi(x);\ x \in [a,b],\ \lim_{y\to x+o} \frac{|\varphi(y)-\varphi(x)|}{|y-x|} = \infty\} .$$

If E has σ-finite linear measure then $\varphi([a,b])$ is a set of constancy.

The proof of this theorem is based upon the following lemma, and its corollaries.

<u>Lemma 1.</u> Let f be continuous on $[a,b]$ and put

$L = \{x \in [a,b);\ f'_+(x) > 0\}$. Then $\lambda(f(L)) \geq f(b)-f(a)$.

(Here $f'_+(x)$ denotes the lower right hand side Dini derivative of f.)

The following simple proof was suggested by Togo Nishiura.

<u>Proof</u>. We can suppose $f(a) < f(b)$. For $y \in [f(a),f(b)]$ we denote

$$\varphi(y) = \max\{x \in [a,b];\ f(x) = y\} .$$

It easily follows from the continuity of f that φ is strictly increasing on $[f(a),f(b)]$. We prove that if φ is differentiable at $y_o < f(b)$ then $y_o \in f(L)$. Let $x_o = \varphi(y_o)$, then $y_o = f(x_o)$, and for $x > x_o$ we have $f(x) > f(x_o)$ and $\varphi(f(x)) \geq x$. This implies

$$f'_+(x_o) = \liminf_{x \to x_o+} \frac{f(x)-f(x_o)}{x-x_o} \geq \liminf_{x \to x_o+} \frac{f(x)-f(x_o)}{\varphi(f(x))-\varphi(f(x_o))} =$$

$$= \frac{1}{\varphi'(y_o)} > 0 .$$

Thus $x_o \in L$ and $y_o = f(x_o) \in f(L)$. Since φ is differentiable at a.e. point of $[f(a),f(b)]$, we have $\lambda(f(L)) \geq f(b)-f(a)$.

<u>Corollary 1.</u> If $\lambda(f(L)) = 0$ then f is decreasing on $[a,b]$.

<u>Proof</u>. Applying Lemma 1 to an arbitrary subinterval $[c,d] \subset [a,b]$ we get $0 \geq (f(d)-f(c))$ and $f(d) \leq f(c)$.

Corollary 2. Let f be continuous and $N = \{x \in (a,b);$ 0 is not a right hand side derived number of f at $x\}$. If $\lambda(f(N)) = 0$, then f is constant on $[a,b]$.

Proof. $N = \{x \in (a,b);\ f'_+(x) > 0$ or $f'^+(x) < 0\}$ and, applying Corollary 1, both f and $-f$ are decreasing on $[a,b]$.

Our next Lemma is an immediate corollary of a theorem of Choquet ([3], p.49.).

Lemma 2. If $A \subset \mathbb{R}^n$ has σ-finite linear measure and $f:A \to \mathbb{R}$ is such that

$$\lim_{\substack{y\to x\\ y\in A}} \frac{|f(y)-f(x)|}{|y-x|} = 0$$

for every $x \in A$ then $\lambda(f(A)) = 0$.

Now we turn to the proof of Theorem 3. Denote $H = \varphi([a,b])$ and let $f:H \to \mathbb{R}$ satisfying (1) for every $x_0 \in H$.

We consider now the function g defined by

$$g(x) = f(\varphi(x)) \qquad (x \in [a,b])$$

and the set

$$N = \{x \in (a,b);\ 0 \text{ is not a righthand side derived number of } g \text{ at } x\}.$$

We prove $\varphi(N) \subset E$. Indeed, let $x \in (a,b)$ be such that $\varphi(x) \notin E$. Then there is a sequence $y_n > x$, $y_n \to x$

such that

$$\left|\frac{\varphi(y_n)-\varphi(x)}{y_n-x}\right|$$

is bounded. Then

$$\left|\frac{g(y_n)-g(x)}{y_n-x}\right| = \left|\frac{f(\varphi(y_n))-f(\varphi(x))}{y_n-x}\right| =$$

$$= \begin{cases} 0 \ , \text{ if } \ \varphi(y_n) = \varphi(x) \\ \dfrac{|f(\varphi(y_n))-f(\varphi(x))|}{|\varphi(y_n)-\varphi(x)|} \cdot \dfrac{|\varphi(y_n)-\varphi(x)|}{|y_n-x|} \ , \text{ if } \varphi(y_n) \neq \varphi(x) \ . \end{cases}$$

Now, by (1), $\dfrac{g(y_n)-g(x)}{y_n-x} \to 0$ and $x \notin N$.

Hence $\varphi(N) \subset E$ and thus

$$\lambda(g(N)) = \lambda(f(\varphi(N))) \leq \lambda(f(E)) = 0$$

by Lemma 2. Therefore, by Corollary 2, g is constant on $[a,b]$ that is, f is constant on $\varphi([a,b])$.

As we mentioned before, Theorem 3 implies Choquet's theorem stating that the graph of any continuous function is a C-curve. Indeed, let f be continuous on $[a,b]$ and let

$$A = \{x \in [a,b];\ \lim_{y \to x+0} \left|\frac{f(y)-f(x)}{y-x}\right| = \infty\} .$$

By a well-known theorem on the contingency of planar sets ([4], p.264.), $E = f(A)$ has σ-finite linear measure. Hence, by Theorem 3, the result follows.

4. Finally we remark that the limit in the definition of H_φ in Theorem 2 cannot be replaced by a one-sided limit. Let $\gamma = \varphi([0,1])$ be the curve given in Theorem 1 and let $P \subset [0,1]$ be a nowhere dense perfect set of positive measure with contiguous intervals (a_j, b_j) $(j=1,2,\dots)$. Let $b_{n_j} = \max\{b_k;\ k < j,\ b_k < b_j\}$ $(j=1,2,\dots)$. We can replace each of the subarcs $\varphi([a_j,b_j])$ by smooth arcs $\gamma_j' = \varphi_j([a_j,b_j])$ running in a small neighbourhood of the arc $\varphi([b_{n_j},b_j])$.

We can define the maps φ_j in such a way that the curve

$$\psi(t) = \begin{cases} \varphi(t) & (t \in Q) \\ \varphi_j(t) & (t \in [a_j,b_j]) \end{cases}$$

has the following properties:

$$\lim_{t \to t_o - o} \left| \frac{\psi(t)-\psi(t_o)}{t-t_o} \right| = \infty$$

for every $t_o \in P$ and

$$\liminf_{t \to t_o + o} \left| \frac{\psi(t)-\psi(t_o)}{t-t_o} \right| < \infty$$

for every $t_o \in [0,1)$. By Theorem 3, the latter condition implies that $\gamma = \psi([0,1])$ is a C-curve.

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