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Vladimír Šverák, MFF UK, Sokolovská 83, 186 00 Prague 8, Czechoslovakia.

Two examples concerning derivatives and M_3-sets.

In his paper [4] Zahorski defined a hierarchy of classes of sets M_i, $i=0,\dots,5$ and a hierarchy of classes of functions $\mathcal{M}_i$, $i=0,\dots,5$. He then demonstrated that the $\mathcal{M}_i$ are closely related to the class of derivatives Δ. In this note we give the following examples concerning the classes M_3 and $\mathcal{M}_3$:

Example 1: There are sets $\emptyset \neq C_1$, $C_2 \subset [0,1]$ which can be separated by a derivative, but cannot be separated by any function of the form $F(x,f(x))$, where f is a bounded derivative and $F:R^2 \to R$ is a continuos function.

Example 2: There is an M_3-set which cannot be written as $f^{-1}(G)$, where f is a derivative and G is an open set.

The first example is connected with the problem of characterizing the smallest "topologically characterizable" system of functions containing bounded derivatives. To be more precise, let us define $\mathcal{F}$ as the smallest system of functions defined on $[0,1]$ possesing the following properties:

(i) $\mathcal{F}$ contains bounded derivatives

(ii) If $f \in \mathcal{F}$ and $\varphi:R \to R$ is a continuos function, then $\varphi \circ f \in \mathcal{F}$

(iii) If $f_n \in \mathcal{F}$ and $f_n \to f$ uniformly on $[0,1]$, then $f \in \mathcal{F}$

It is not difficult to show that $\mathcal{F}$ admits a characterization in terms of pseudouniformities. (Use, for example, Theorem 3.1. from [3] and the Darboux property of derivatives.) Clearly $\mathcal{F} \subset b\mathcal{M}_3$, where $b\mathcal{M}_3$ denotes the system of bounded $\mathcal{M}_3$ -functions defined on $[0,1]$. The example 1 shows that $\mathcal{F} \neq b\mathcal{M}_3$, because the sets C_1 , C_2 can be separated by a function from $b\mathcal{M}_3$ and cannot be separated by any function from $\mathcal{F}$. This follows from the fact that the closure of the set $Q = \{ g: [0,1] \to R$, $g(x) = F(x,f(x))$, $F:R^2 \to R$ continuos, f is a bounded derivative$\}$ in the topology of uniform convergence on $[0,1]$ contains $\mathcal{F}$. We remark that we are able to prove $\mathcal{F} - Q \neq \emptyset$.

The second example shows that not every M_3-set can be written as $\{x, \varphi(f(x)) > 0\}$, where f is a derivative and φ is a continuos function. It is not known whether every M_3-subset of $[0,1]$ can be written as $\{x, g(x) > 0\}$ with $g \in \mathcal{F}$.

We will use the following notation. If $x \in R$ and $A \subset R$ is a measurable set, we denote by $\underline{d}(x,A)$ the lower density of the set A at x . If $f:[0,1] \to R$ is a function and $a \in R$, we denote by $\{f > a\}$ the set $\{x \in [0,1]$; $f(x) > a\}$. The distance between two sets A and $B \subset R$ is denoted by dist(A,B).

To begin with our examples, let us prove the following

<u>Lemma 1</u>: Let $\emptyset = F_{-1} \neq F_0 \subset\cdot F_1 \subset\cdot F_2 \ldots \subset\cdot F_{2n+1}$ be closed, nowhere dense sets. (Here $A \subset\cdot B$ means, as usual, that $A \subset B$ and every point of A is a point of density of the set B.)

Put $H_1 = \bigcup_{k=0}^{n} (F_{2k} - F_{2k-1})$, $H_2 = \bigcup_{k=0}^{n} (F_{2k+1} - F_{2k})$.
If f is a bounded derivative such that $\mathrm{dist}(f(H_1), f(H_2)) \geq \delta > 0$, then there are points $a_k \in F_k - F_{k-1}$ such that, for any $k=0,\ldots, 2n+1$, a_k is an isolated point of the set $\{f > f(a_k) - \delta\} \cap F_k$ from the right and $f(a_k) \geq f(a_o) + k\delta$.

Proof: We shall define the sequence $a_o, \ldots, a_{2n+1}$ by induction. For a_o take an arbitrary point of F_o isolated from the right. Suppose the points $a_i \in F_i - F_{i-1}$, $0 \leq i \leq k \leq 2n$ have already been defined. Find $\eta > 0$ such that

(1) $\quad (a_k, a_k+\eta) \cap \{f > f(a_k) - \delta\} \cap F_k = \emptyset$

and put $M = (a_k, a_k+\eta) \cap \{f > f(a_k) - \delta\} \cap F_{k+1}$. As f is a bounded derivative and $\underline{d}(a_k, F_{k+1}) = 1$, the set M is nonempty. As (1) holds, we have $M \subset F_{k+1} - F_k$ which implies that $|f(x) - f(a_k)| \geq \delta$ for any $x \in M$. Hence $M = (a_k, a_k+\eta) \cap \{f \geq f(a_k) + \delta\} \cap F_{k+1}$. We conclude that M is a nonempty F_σ and G_δ subset of $F_{k+1} - F_k$. Thus there is a point y of M isolated from the right. Put $a_{k+1} = y$. Clearly $\{f > f(a_{k+1}) - \delta\} \cap F_{k+1} \cap (a_k, a_k+\eta) \subset M$, from which it follows that a_{k+1} is an isolated point of the set $\{f > f(a_{k+1}) - \delta\} \cap F_{k+1}$ from the right. From the definition of a_{k+1} we see that $f(a_{k+1}) \geq f(a_k) + \delta \geq f(a_o) + (k+1)\delta$, which shows that a_{k+1} satisfies our conditions.

The construction of the sets C_1, C_2.

For $n = 1,2, \ldots$ let I_n denote the interval $(1/2+1/(n+2), 1/2+1/(n+1))$. Find nowhere dense closed sets

$\emptyset = F_{-1}^{n} \neq F_{o}^{n} \subset \cdot F_{1}^{n} \subset \cdot \ \ldots \ \subset \cdot F_{2n+1}^{n} \subset I_{n}$.

Let $H_1^n = \bigcup_{k=0}^{n} (F_{2k}^n - F_{2k-1}^n)$, $H_2^n = \bigcup_{k=0}^{n} (F_{2k+1} - F_{2k})$

and put $C_1 = \bigcup_{n=1}^{\infty} H_1^n$, $C_2 = \bigcup_{n=1}^{\infty} H_2^n$.

If f is a bounded derivative, it follows from Lemma 1 that, for any $\eta > 0$, we have

$\mathrm{dist}(f(C_1 \cap (1/2, 1/2+\eta)) , f(C_2 \cap (1/2, 1/2+\eta))) = 0$.

Hence the sets C_1, C_2 cannot be separated by any function of the form $F(x, f(x))$ with $F: R^2 \to R$ continuos.

On the other hand, from [1] (Theorem 4.5.) it follows that C_1, C_2 can be separated by a derivative, hence also by a bounded $\mathcal{M}_3$-function.

To begin example 2, we prove the following

<u>Lemma 2</u>: Let $f: [0,1] \to R$ be a derivative and $J = (a,b) \subset R$ an open interval such that $E = f^{-1}(J) \neq \emptyset$. Then there is a portion of E which belongs to the class M_4 .

<u>Proof</u>: Let $x_o \in \bar{E} \cap (0,1)$ be a point of continuity of the function $f / \bar{E}$. Suppose, for example, $f(x_o) < b' < b$. There is an open interval $I \subset (0,1)$ such that $x_o \in I$ and $f(x) < b'$ for any $x \in \bar{E} \cap I$. As f is a Darboux function, we have $f(x) < b'$ for any $x \in I$. We conclude also that $I \cap E = \{ f > a \} \cap I$ so that $I \cap E$ is a nonempty M_4-set. If $f(x_o) = b$, consider the function $g(x) = a+b-f(x)$.

(Remark: We are able to show that in fact $E \in M_3^*$.(For the definition of M_3^* see [2] .) But here we shall not need this fact.)

Corollary: Let $G=\bigcup_n (a_n,b_n)\subset R$ be an open set and $f:[0,1]\to R$ a derivative. If $E = f^{-1}(G) \neq \emptyset$, then E contains a non-empty M_4-set E_1 such that $E-E_1$ is an F_δ-set.

Proof: Choose k such that $E_0 = f^{-1}(a_k,b_k) \neq \emptyset$. Let $E_1=E_0\cap I\neq\emptyset$ be a portion of E_0 belonging to the class M_4. We can write $E-E_1 = \bigcup_{n\neq k} f^{-1}(a_n,b_n)\cup(f^{-1}(a_k,b_k) -I)$ and we see that $E-E_1$ is an F_δ-set.

Example 2: Let E be an M_3-set such that $\emptyset \neq E = \bigcup_n F_n$, with $F_n\subset F_{n+1}$ closed nowhere dense sets , and $\overline{d}(x,(R-F_n)\cap E) = 0$ for any $x\in F_n$, n=1,2, ... It is easy to see that the set E possesses the following property: If $A\subset E$ is an M_4-set, then, for each n, $A\cap F_n$ is also an M_4-set (which can, of course, be empty) . Now let us suppose that $E=f^{-1}(G)$ for some derivative f and open set G. Find the set E_1 from the corollary of Lemma 2. As $\emptyset \neq E_1\subset E$, $E_1\cap F_n \neq \emptyset$ for some n . But the set $E_1\cap F_n$ cannot belong to the class M_4 . Indeed, it is clearly F_δ and, because we can write $E_1\cap F_n = F_n\cap(R -(E-E_1))$, we see that it is also G_δ. Furthemore, $E_1\cap F_n\subset F_n$ and F_n is nowhere dense. We conclude that $E_1\cap F_n$ contains a point isolated from the right and this is the required contradiction.

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