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INTEGRABILITY CONDITIONS FOR APPROXIMATE DERIVATIVES

1. Introduction:

In light of Richard Fleissner's untimely death, the authors, who feel that the results have been influenced by Dick's work, dedicate this paper to him. It can be viewed as a natural next step based on the results of [3] and [4]. In [3] conditions were examined under which an approximate derivative became Lebesgue integrable. It was found that it is both necessary and sufficient to consider the Lebesgue integrability of the function F^*. This F^*, as in [4], equals F' wherever it exists and 0 elsewhere. Similarly, F^*_{ap} will denote F'_{ap} when it exists and 0 elsewhere. Here both the wide and restricted Denjoy integrals will replace the Lebesgue integral. For the restricted $\mathcal{D}$ integral the situation is found to parallel results of [3], but for the wide sense $\mathcal{D}$ integral the results are negative.

2. The restricted sense Denjoy integral, $\mathcal{D}_*$.

Preliminaries:

Definition. A function $F: [0,1] \to R$ is called Baire*1 if

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there is a sequence of closed sets X_n such that $\bigcup_{n=1}^{\infty} X_n = [0,1]$ and $F|X_n$ is continuous for each n.

<u>Theorem</u> [3]: Let F: $[0,1] \to R$ be Baire*1 Darboux. Let $U(F) = \text{interior } \{x: F \text{ is continuous at } x\}$. Suppose that F has Lusin property (N) on U(F). Let $P = \{x: F' \text{ exists at } x \text{ and } F'(x) > 0\} \cap U(F)$. Then F is absolutely continuous on [0,1] if and only if F' is Lebesgue integrable over P.

<u>Corollary</u>: Let F: $[0,1] \to R$ and U(F) be as above. If F has (N) on U(F) and $F' = 0$ almost everywhere where F' exists, then F is a constant.

We now have:

<u>Theorem</u>: Let F: $[0,1] \to R$ be approximately differentiable and let

$$F^* = \begin{cases} F' & \text{where } F' \text{ exists} \\ 0 & \text{elsewhere} \end{cases}$$

Then, if F^* is $\mathcal{D}_*$ integrable, F'_{ap} is $\mathcal{D}_*$ integrable.

<u>Proof</u>. Let G be a $\mathcal{D}_*$ primitive of F^*. Let $H = F - G$. It is known that H is Baire*1, Darboux and has (N) (see [3], p. 261). Let x be a point where H' exists and at the same time both G' exists and $G' = F^*$. Then F' exists at x and hence $G' = F'$ and $H' = 0$.

Consequently, almost everywhere where H' exists it is equal to zero. By the corollary stated above, the function H must be a constant. This yields that F itself is a $\mathcal{D}_*$ primitive of F*. This, in turn, requires that F' exists almost everywhere i.e., $F'_{ap} = F^*$ a.e. So trivially F'_{ap} is $\mathcal{D}_*$ integrable. By similar arguments it can be proved that if F'_{ap} is $\mathcal{D}_*$ integrable, then F is a $\mathcal{D}_*$ primitive and $F'_{ap} = F^*$ almost everywhere. This means that

$$F'_{ap} \text{ is } \mathcal{D}_* \text{ integrable} \Longleftrightarrow F^* \text{ is } \mathcal{D}_* \text{ integrable.}$$

It is clear that the above proof does not essentially depend on F being approximately differentiable. For example, if F were merely assumed to be selectively differentiable, [5], all the necessary conditions would be available to yield F'_s is $\mathcal{D}_*$ integrable $\longleftrightarrow$ F* is $\mathcal{D}_*$ integrable.

3. The wide-sense Denjoy integral, $\mathcal{D}$

Let F: [0,1] → R be approximately differentiable for all x in [a,b]. We wish to examine if the $\mathcal{D}$ integrability of F* will imply that F'_{ap} is $\mathcal{D}$ integrable. If F is approximately differentiable and if F'_{ap} is $\mathcal{D}$ integrable, then F is a $\mathcal{D}$ primitive of F'_{ap} (this follows from the Corollary); in particular F is continuous. Moreover, if F is continuous and approximately differentiable, then it is a $\mathcal{D}$ primitive of F'_{ap} (see [3], p. 261). It is thus clear that the problem reduces to determining whether the $\mathcal{D}$ integrability of F* implies F is continuous. With that insight it is perhaps not surprising that a counterexample exists. We will present such an example. Further we will present counterexamples for other possible conjectures.

Lemma 1. Let $[c,d]$ be an interval and α,β two real numbers. Let E be any perfect nowhere dense but metrically dense subset of $[c,d]$. (By metrically dense, we mean that for each open interval I, $I \cap E \neq \phi$ implies $|I \cap E| > 0$, where $|\ |$ denotes Lebesgue measure.) Then there is a differentiable function $g\colon [c,d] \to R$ such that

g is monotone,

$g(c) = \alpha$, $g(d) = \beta$,

$g'(c) = g'(d) = 0$,

$g' \equiv 0$ on any interval contiguous to E,

$g' \not\equiv 0$ on any open interval intersecting E.

Proof. This is a known result which can be obtained, for example, by an application of Theorem 6.8 in [2] (p. 35).
Making use of Lemma 1, repeatedly, we define a function F_1. Let I_n be any sequence of closed intervals $[a_n,b_n]$ with $0 < b_{n+1} < a_n < b_n$, $a_n \to 0$, and $\bigcup_{n=1}^{\infty} I_n$ having density 0 at 0. For each n, in the left half of I_n we pick a nowhere dense, perfect, metrically dense set E_n containing a_n. For $[c,d] = [a_n,(a_n + b_n)/2]$, $\alpha = 0$, $\beta = 1$, and $E = E_n$ we apply

Lemma 1, to get a differentiable function g_n with the stated properties. We extend the definition of g_n to the right half of I_n by reflecting about the line $x = (a_n + b_n)/2$, and let E_n^* denote $E_n \cup$ (its reflection about the line $x = (a_n + b_n)/2$). We define

$$F_1(x) = \begin{cases} g_n(x) \text{ if } x \in I_n \quad n = 1,2, \ldots \\ 0 \text{ otherwise in } [0,1]. \end{cases}$$

Then F_1 is differentiable on $(0,1]$ and approximately differentiable at 0, and F_1 is not continuous at $x = 0$.

Lemma 2. Let E be a perfect, metrically dense, nowhere dense subset of $[c,d]$ containing c and d. There is a continuous function $g\colon [c,d] \to R$ such that

$g \equiv 0$ over E,

g is approximately differentiable for all x in $[c,d]$,

$g'_{ap} \equiv 0$ over E,

g is not differentiable at any x in E,

g is differentiable on $[c,d] \setminus E$,

$\sup\{|g(x)| : x \in [c,d]\} \le d-c$.

Proof. Let U denote the complement of E relative to $[c,d]$. For convenience sake we arrange the components of U as in the construction of the Cantor set.

That is, for each n we select 2^{n-1} components of U, I_{ni}, $i = 1, 2, \ldots, 2^{n-1}$, as follows.

Let $G_{11} = [c,d]$.

Let I_{11} be any component of U with $|I_{11}| = \sup\{|J|:$ J is a component of U$\}$.

Let the two closed intervals which compose the complement of I_{11}, relative to [c,d], be designated as G_{21} and G_{22}.

Let I_{21} be any component of U with $|I_{21}| = \sup\{|J|:$ J is a component of $U \cap G_{21}\}$, $I_{21} \subset G_{21}$.

Let I_{22} be a similar component of U in G_{22}.

Let the four remaining intervals be designated as $G_{31}, \ldots, G_{34}$.

Proceed inductively to label G_{ni}, $i = 1, \ldots 2^{n-1}$, and pick components $I_{ni} \subset G_{ni}$. This process will arrange all the components of U, and for each x in E there will be a unique nested sequence of intervals G_{ni_n} with $\bigcap_{n=1}^{\infty} G_{ni_n} = \{x\}$, and $|G_{ni_n}| \to 0$ as $n \to +\infty$. Inside each I_{ni}, we select an open interval w_{ni}, centered about the midpoint of I_{ni}, in such a fashion that $\bigcup_{ni} w_{ni}$ has density zero at each point of the E.

We are ready to define $g: [c,d] \to R$.

For each n,i fixed let [u,v] = closure of w_{ni}. Let g be any differentiable function with $g(u) = g'(u) = g'(v) = g(v) = 0$, $g((u+v)/2) = |G_{ni}|$, and g strictly increasing from u to the midpoint of [u,v] and strictly decreasing from the midpoint to v.

Let $g(x) = 0$ for all other x in [c,d].

Of the properties stated for g(x) in the conclusion of the lemma, it is only the nondifferentiability at points of E that may not be immediately clear. Let x belong to E and let G_{ni_n} be the unique sequence such that $\bigcap_{n=1}^{\infty} G_{ni_n} = \{x\}$. Let m_n be the midpoint of w_{ni_n}. We have $m_n \in G_{ni_n}$ so $m_n \to x$ as $n \to +\infty$ and

$$\left|\frac{g(x) - g(m_n)}{x - m_n}\right| \geq \left|\frac{G_{mi_n}}{x-m_n}\right| \geq 1$$

Therefore g has a derived number of absolute value at least 1 at x. But, in addition, since E is perfect and $g \equiv 0$ over E, 0 is a derived number of g at x. Therefore g is not differentiable at x.

<u>Example</u> <u>1</u>. There is an approximately differentiable F such that F* is $\mathcal{D}$ integrable and F'_{ap} is not $\mathcal{D}$ integrable.

We now combine Lemma 2 with our construction of F_1 above. For each n, let $I_n = [a_n,b_n]$ be the intervals in the construction of F_1 and let g_n be the function of Lemma 2 for $[c,d] = I_n$ and $E = E_n^*$.

$$\text{Let } F_2(x) = \begin{cases} g_n(x) \text{ if } x \in I_n & n = 1,2,\ldots \\ 0 \text{ elsewhere in } [0,1]. \end{cases}$$

Then, because $|g_n(x)| \leq |I_n|$, we have that F_2 is continuous on [0,1], and approximately differentiable. Hence $(F_2)'_{ap}$ is $\mathcal{D}$ integrable.

Next, consider $F = F_1 + F_2$: $[0,1] \to R$. F is not continuous but is approximately differentiable. A moment of checking verifies that $F^* = (F_2)'_{ap}$. From this, we can conclude that the $\mathcal{D}$ integrability of F^* will not imply the $\mathcal{D}$ integrability of F'_{ap} unlike the situation for the L integral.

In another sense, F^* is near to being the proper object for determining the $\mathcal{D}$ integrability of F'_{ap}. More precisely, again let F: $[0,1] \to R$ be approximately differentiable and let ΔF = interior {x: F is differentiable at x}. It is known (see [1], Theorems 2 and 3) that there is a differentiable function H such that $H' = F'_{ap}$ over $[0,1] \setminus \Delta F$. Consider then $F - H$ and the corresponding $(F-H)^*$. It is easy to see that

i) F'_{ap} is $\mathcal{D}$ integrable $\Leftrightarrow (F-H)'_{ap}$ is $\mathcal{D}$ integrable,

ii) $(F-H)'_{ap} = (F-H)^*$.

Hence F'_{ap} is $\mathcal{D}$ integrable $\Leftrightarrow (F-H)^*$ is $\mathcal{D}$ integrable.

We now include some examples which point out the boundaries of the associated problems.

If F is continuous and approximately differentiable, then F'_{ap} is $\mathcal{D}$ integrable but F^* need not be $\mathcal{D}$ integrable. For let $F_1(x) = x^2\sin(x^{-4})$, with $F_1(0) = 0$. Let E be any metrically dense subset of $\{x: F_1'(x) \geq 0\}$ with the additional property that the integral of $F_1'(x)$ over E is infinite. Then if $F_2(x)$ is 0 on E and ACG and $F_2'(x)$ does not exist if and only if $x \in E$, then $F = F_1 + F_2$ is ACG but F^* is not $\mathcal{D}$ integrable. The construction of such an F_2 is given in [6], p. 224. This is because $F'_{ap} - F^* = F_1' \chi_E$ which is not $\mathcal{D}$ integrable.

From [4] and [6] it follows that if F is a continuous function satisfying (N) and if F^* is $\mathcal{D}_*$ integrable, then F is a $\mathcal{D}_*$ primitive of F', which exists almost everywhere. We finish this paper with a preliminary example and an example whose purpose is to show that the analogous result for the $\mathcal{D}$ integral does not hold; there is a continuous function G, satisfying (N), and approximately differentiable a.e. such that G^* is $\mathcal{D}$ integrable and G'_{ap} is not $\mathcal{D}$ integrable.

Preliminary Example. There exists a continuous function F on [0,1/2] which is differentiable almost everywhere, has (N) and is of unbounded variation on every subinterval of [0,1/2].

$$\text{Let } g_n(x) = \begin{cases} x^2 \sin^2(\Pi x^{-4}) & \text{if } 0 < |x| \le 2^{-n}, \\ 0 & \text{otherwise} \end{cases}$$

Then g_n is differentiable and of unbounded variation in every neighborhood of 0. Let $\{r_k\}_{k-1}^{\infty}$ be an enumeration of the rational numbers in (0,1/2), $\{n_k\}$ be an increasing sequence of natural numbers, $f_k = g_{n_k}(x-r_n)$, $F_k = \sum_{n=1}^{k} f_n$ and $F = \lim_{k\to\infty} F_k$. Then F is continuous.

Further let $I_k = [r_k - 2^{-n_k}, r_k + 2^{-n_k}]$,

$J_k = [r_k - (2^{-n_k}\cdot n_k^2), r_k + (2^{-n_k}\cdot n_k^2)]$, and $E_N = \bigcup_{k=N}^{\infty} J_k$, and $E = \bigcap_{N=1}^{\infty} E_N$.

Then $|E| = 0$.

Claim A. If x does not belong to E_N, then $F'(x)$ exists and equals $F_N'(x)$. Thus $F'(x)$ exists almost everywhere.

Proof of Claim A. Let $\varepsilon > 0$: Choose $M > N$ so that $\sum_{k=M}^{\infty} \frac{1}{n_k^2-1} < \varepsilon$ and set $\delta = (n_M^2-1)2^{-n_M}$. Let $0 < |h| < \delta$.

If $N \le k \le M$, $f_k(x+h) = f_k(x) = 0$. If $k \ge M$,

$$|f_k(x+h)-f_k(x)| = \begin{cases} < 2^{-n_k} & \text{if } x+h \text{ is in } I_k, \text{ and then } |h| > (n_k^2-1)2^{-n_k}, \\ 0 & \text{if } x+h \text{ is not in } I_k. \end{cases}$$

Then

$$\left|\frac{F(x+h)-F(x)}{h} - \frac{F_N(x+h)-F_N(x)}{h}\right| = \left|\sum_{k=M}^{\infty} \frac{f_k(x+h)-f_k(x)}{h}\right|$$
$$\le \sum_{k=M}\left(\frac{2^{-n_k}}{(n_k^2-1)\cdot 2^{-n_k}}\right) < \varepsilon .$$

Since ε is arbitrary, the claim A is established.

Claim B. Provided n_k are chosen so that $O(F_{k-1},J_k) < 2^{-k}$, F satisfies (N).

Proof of Claim B. It suffices to show that $|F(E)| = 0$ because F is differentiable on E^c and thus has (N) on E^c. Since $|f_k(x)| < 2^{-k}$ for all k and $E \subset \bigcup_{n=N}^{\infty} J_n$ for all N, it follows that

$$|F(E)| < \sum_{n=N}^{\infty} |F(J_n)| = \sum_{n=N}^{\infty} O(F,J_n)$$

$$\leq \sum_{n=N}^{\infty} \overline{\lim_k}\, O(F_k, J_n)$$

$$\leq \sum_{n=N}^{\infty} O(F_{n-1},J_n) + \sum_{k=n}^{\infty}(O(f_k,J_n))$$

$$\leq \sum_{n=N}^{\infty} (2^{-n} + \sum_{k=n}^{\infty} 2^{-k})$$

$$= (3/2)(2^{-N+1}) = 3 \cdot 2^{-N}$$

Since N is arbitrary it follows that $|F(E)| = 0$.

Claim C. If n_k is chosen so that for each k, J_k does not contain $r_1, r_2, \ldots, r_{k-1}$, and $\int_{J_k} |F'_{k-1}(x)|dx < 2^{-k}$, then F is of unbounded variation on every neighborhood of $(0,1/2)$.

Proof of Claim C. We first note that since

$$O(F_{k-1},J_n) \leq \mathrm{Var}(F_{k-1},J_k) = \int_{J_k} |F'_{k-1}(x)|dx < 2^{-k},$$

F also satisfies (N). Furthermore, such n_k can be chosen inductively because F'_k is Lebesgue integrable on each chosen interval which does not contain $r_1, \ldots, r_{k-1}$ and hence

$$\lim_{\substack{|I|\to 0 \\ r_k \varepsilon I}} \left(\int_{|I|} |F'_k(x)|dx \right) = 0.$$

Given an interval I, it suffices to show that

$$\int_I |F'(x)|dx = \infty.$$

But $\int_I |F'(x)|dx \geq \int_{I\setminus \bigcup_{n>N} J_n} |F'(x)|dx$ and if N is chosen so that $r_N \in I$,

$$\int_{I\setminus \bigcup_{n>N} J_n} |F'(x)|dx = \int_{I\setminus \bigcup_{n>N} J_n} |F_N'(x)|dx = +\infty .$$

This completes the Preliminary Example.

Example 2. There is a function G continuous on [0,1] which satisfies (N) and is approximately differentiable a.e. such that G^* is $\mathcal{D}$ integrable, and G'_{ap} is not $\mathcal{D}$ integrable.

Construction. Let P be a perfect, metrically dense subset of [0,1] with $|P| = 1/2$, P nowhere dense, and $\{0,1\} \subset P$. Let $b(x) = |P\cap[0,x]|$, $0\leq x\leq 1$. Then h(x) is monotone nondecreasing, $h'(x) = 1$ for almost all x in P and $h'(x) = 0$ at every point of P^c. Let F be a continuous function, given in the preliminary example, which satisfies (N) and is not of bounded variation on any interval. Then Foh(x) has derivative 0 at each x in P and is differentiable at almost all x in P but is not of bounded variation on any portion of P. Let g(x) be the function defined in [7,p. 224) which is ACG 0 on P, and not differentiable at any x in P. Let G(x) = g(x) + Foh(x). Then, at almost every point x of P, G'(x) does not exist. Thus $G'(x) = g^*(x) = g^*_{ap}(x)$ is $\mathcal{D}$ integrable. But G(x) is not of bounded variation on any portion of P. Therefore, since $G^*_{ap}(x) = g'(x)$ in P^c and $G^*_{ap} = F' \circ h$ in P, it is not $\mathcal{D}$ integrable. For if it were, G would have to be its $\mathcal{D}$ integral by [4] and G would be ACG on P, a contradiction.

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