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PERRON-STIELTJES INTEGRABILITY WITH RESPECT TO GAP FUNCTIONS

The advantage of Perron integration over Lebesgue integration is that derivatives are always integrable. However, its disadvantage is that one has little feeling for what it means for a function to be Perron integrable.

Using the generalized Riemann approach we investigate conditions under which a function is Lebesgue-Stieltjes and/or Perron-Stieltjes integrable with respect to a measure consisting of countably many point masses. As often happens, the Lebesgue case is simple and essentially corresponds to absolute convergence of a series, while the Perron case involves not only conditional convergence of a series but also the topology of the set of points at which masses are placed.

One can and often does think of integrability questions in terms of more-or-less discrete "chunks," which correspond to the contributions from the point masses. In this way we hope the results here shed some light on the question of Perron integrability in general.

The specific problems in this paper and in fact the entire generalized Riemann approach to integration was brought to the author's attention in a seminar at the University of Petroleum and Minerals conducted by W. Pfeffer, to whom the author is grateful for several enlightening conversations. The author also thanks the referee for suggesting many improvements in the presentation.

We give below a very brief outline of the generalized Riemann approach to the Perron-Stieltjes and Lebesgue-Stieltjes integrals. We follow the terminology and notation of [3] with some modifications for our special purposes. The reader should consult [3] for more details.

By an interval J we will always mean a nondegenerate closed bounded interval of real numbers. All functions we deal with will be real valued. In this paper α will always denote a nondecreasing function whose domain is the real number line. For such an α we set $\alpha([a, b]) = \alpha(b) - \alpha(a)$. The support of α is defined to be $\{x|\alpha([a, b]) > 0$ whenever $a < x < b\}$.

Our primary concern will be with Perron integration, so our definition of "partition" will be slightly nonstandard. By a partition P of an interval J we mean a collection $\{I_1, I_2, \ldots, I_n; x_1, x_2, \ldots, x_n\}$, where $I_1, I_2, \ldots, I_n$ are nonoverlapping (their interiors are pairwise disjoint) intervals whose union is J and x_k is a point of the interval I_k for each $k = 1, 2, \ldots, n$. We say that the x_k's are points of P and that x_k is the point of P corresponding to I_k. If we require only that each point of P belongs to J and not necessarily to its corresponding interval, then we say that P is a Lebesgue partition.

If δ is a positive function on the interval J then we say that a partition, Lebesgue or not, is δ-fine if I_k is contained in $(x_k - \delta(x_k), x_k + \delta(x_k))$ for each k. In fact, we use the term "δ-fine" to refer to any (finite or infinite) collection $\{J_k; x_k\}$ of intervals J_k and corresponding points x_k for which $J_k \subset (x_k - \delta(x_k), x_k + \delta(x_k))$. We should note that for any positive function δ there are δ-fine partitions, even such that each point is in the interior of its corresponding interval.

See [3, p. 4].

Let f be any function on an interval J and let P be a Lebesgue partition of J, say $P = \{I_1, I_2, \ldots, I_n; x_1, x_2, \ldots, x_n\}$. We denote the sum $\sum_{k=1}^{n} f(x_k)\,\alpha(I_k)$ by $\sigma(f, P, \alpha)$ or just $\sigma(f, P)$ when no confusion results. If there is a number A such that for each positive ε there is a positive function δ on J for which $|\sigma(f, P) - A| < \varepsilon$ whenever P is a δ-fine Lebesgue partition of J then we say that f is Lebesgue-Stieltjes integrable (or just L-integrable) with respect to α on J. If we require only that $\sigma(f, P)$ is near A for P a δ-fine partition of J, then we say that f is Perron-Stieltjes integrable (or just P-integrable) with respect to α on J. We denote A by $L\int_J f d\alpha$ in the Lebesgue case and by $P\int_J f d\alpha$ in the Perron case. This definition of Lebesgue-Stieltjes integral does not agree exactly with that in [4] but does agree with [3].

We need to make clear our conventions regarding sequences and countable sets. A countable set will be a set which is finite or countably infinite but not empty. Consistent with this, a sequence is a function whose domain is either $\{1, 2, \ldots, n\}$ for some positive integer n or the entire set of positive integers. We use the usual notation $\{s_k\}$ for a sequence whose value at k is s_k. An indexing of a set Q is a one-to-one sequence whose image is exactly Q. Sums will be over relevant indices.

Now we can define a gap function. Let Q be a countable set of points in the interior of an interval I. Let α' be a function from Q into the nonnegative real numbers such that $\sum\{\alpha'(y)\,|\,y \in Q\}$ converges. A gap function determined by Q and α' is then any function whose value at a point x not in Q is given by $\alpha(x) = \sum\{\alpha'(y) \mid y < x$ and $y \in Q\}$ and whose value at a point x belonging to Q is between $\sum\{\alpha'(y) | y < x$ and $y \in Q\}$ and $\sum\{\alpha'(y) | y \leq x$ and $y \in Q\}$ inclusive.

BASIC RESULTS AND DEFINITIONS. In this section we present some basic though elementary results concerning integration with respect to gap functions and define some properties which are motivated by these results. The first result comes from [3], to which the reader is referred for a proof.

Theorem 1. Let α be a gap function determined by Q and α', where Q lies in the interior of the interval I. Suppose that $\{s_k\}$ is any indexing of the points of Q. Let f be any function on I. Then f is L-integrable with respect to α if and only if $\sum f(s_k)\alpha'(s_k)$ is absolutely convergent and in this case $L\int_I f d\alpha = \sum f(s_k)\alpha'(s_k)$.

Let α be a gap function determined by Q and α', where Q is contained in the interior of the interval I. Suppose that f and g are functions on I which agree on Q. Then $f-g$ is zero on Q and so by Theorem 1 $f-g$ is L-integrable and hence P-integrable with respect to α on I. Consequently f is L-integrable (P-integrable) with respect to α on I if and only if g is L-integrable (P-integrable) with respect to α on I. Thus we can and do assume throughout that functions have values 0 off the set Q in question.

The proof of the next theorem uses techniques of [3] and is not difficult, so we omit it and only state the result. We say that a sequence $\{s_n\}$ is monotone if either $s_n > s_{n+1}$ for all relevant n or $s_n < s_{n+1}$ for all relevant n.

Theorem 2. Let α be a gap function determined by Q and α'. Suppose that the monotone sequence $\{s_n\}$ is an indexing of Q and that Q lies in the interior of the interval I. A function f is P-integrable with respect to α on I if and only if $\Sigma f(s_n)\alpha'(s_n)$ converges and in this case $P\int_I f d\alpha = \Sigma f(s_n)\alpha'(s_n)$.

If the set Q used in the definition of a gap function α has just finitely many limit points, then Q can be decomposed into finitely many monotone sequences, so Theorem 2 furnishes a way of determining integrability of a function with respect to such an α. We would like to relax the condition imposed on the set Q and obtain characterizations of integrability for the resulting gap functions. Theorem 2 motivates the following definition.

Let α be a gap function determined by Q and α'. We say that a function f defined on Q has Property S with respect to α if each monotone sequence in Q is a subsequence of a monotone sequence $\{x_n\}$ in Q for which $\Sigma f(x_j)\alpha'(x_j)$ converges.

Theorem 3. Let α be a gap function determined by Q and α'. Suppose the set Q has just finitely many limit points and is contained in the interior of the interval I. Then a function f is P-integrable with respect to α on I if and only if f has Property S with respect to α.

Proof. That Property S implies P-integrability when Q has just finitely many limit points follows quickly from Theorem 2 and the subsequent remarks. The necessity of Property S follows from Theorems 4 and 5 in the next section, which depend on the development of another idea and a bit of general machinery.

It would be very satisfying if Theorem 3 remained true when the conditions on the set Q are relaxed, but an example to follow destroys hope of this. We must develop some machinery before we can easily present this example so first we define another property modelled on Property S but incorporating a type of δ-fine partition as well.

We extend the meaning of "monotone" by saying that a sequence $\{I_n\}$ of nonoverlapping intervals is monotone if the sequence of left endpoints of the intervals is monotone. Now suppose the function α (not necessarily a gap function, but nondecreasing) is given. Then we say that a function f on the interval I has Property Σ with respect to α on I if whenever δ is a positive function and $\{I_n\}$ is a monotone sequence of subintervals of I, then there is another monotone sequence of subintervals $\{J_n\}$ and a sequence of points $\{x_n\}$ such that $x_n \in J_n$, $\{J_n; x_n\}$ is δ-fine, each I_n is a union of a subcollection of $\{J_k\}$ and $\Sigma f(x_n)\alpha(J_n)$ converges.

PROPERTY S AND PROPERTY Σ, GENERALITIES. We now begin a study of these properties and how they are related to the integrability of a function. Except where explicitly stated otherwise, α need only be nondecreasing and need not be a gap function.

Lemma 1. Let f be P-integrable with respect to α on the interval I and let b be any point of I. Then for each positive ε there is a positive function δ on I such that whenever P is a δ-fine partition of a subinterval of I lying in $(b - \delta(b), b + \delta(b)) - \{b\}$, then

$|\sigma(f, P)| < \varepsilon$.

Proof. Let b be a point of the interval $I = [a, c]$ and let ε be positive. We deal with $(b - \delta(b), b)$, the other side of b then being treated similarly. Since f is P-integrable with respect to α there is a positive function δ on I such that if P and P' are δ-fine partitions of I then $|\sigma(f, P) - \sigma(f, P')| < \varepsilon/2$. We may assume that $\delta(b)$ is so small that $|f(b)|\alpha([b - \delta(b), s]) < \varepsilon/2$ whenever $s \in (b - \delta(b), b)$.

Let P be a δ-fine partition of an interval of $[r, s]$ in $[b - \delta(b), b)$. Use the interval $[s, b]$ with corresponding point b, a δ-fine partition of $[a, r]$ and a δ-fine partition of $[b, c]$ together with P to get a δ-fine partition Q of I.

Use the same partitions of $[a, r]$ and $[b, c]$ and the interval $[r, b]$ with corresponding point b to get a δ-fine partition Q' of I. Now $|\sigma(f, Q) - \sigma(f, Q')| < \varepsilon/2$ and $|\sigma(f, Q) - \sigma(f, Q')| =$ $|\sigma(f, P) + f(b)\alpha([s, b]) - f(b)\alpha([r, b])| = |\sigma(f, P) - f(b)\alpha([r, s])|$. Since $|f(b)\alpha([r, s])| < \varepsilon/2$, we obtain $|\sigma(f, P)| < \varepsilon$.

Theorem 4. If the function f is P-integrable on the interval I with respect to α, then f has Property Σ with respect to α on I.

Proof. Let $\{I_n\}$ be a monotone sequence of subintervals of I. We will consider the case in which the left endpoints of the I_n's increase to limit b, the decreasing case being exactly similar. Without loss of generality we suppose that the union of the I_n's is $[r, b)$ for some r, since otherwise we can add the omitted intervals in their proper places in the sequence.

Lemma 1 implies that for each positive integer n there is a positive function δ_n such that if P is a δ_n-fine partition of an interval in $I \cap [b - \delta_n(b), b)$, then $|\sigma(f, P, \alpha)| < 1/n$. We may suppose that $\delta_1 > \delta_2 > \ldots$ and that $\{\delta_n(b)\}$ converges to 0.

Consider the interval I_j. If n is the smallest integer for which I_j misses $[b - \delta_n(b), b)$, then let P_j be a δ_n-fine partition for I_j. If we list the intervals of $P_1, P_2, \ldots$ in increasing order then we have a monotone sequence $\{J_n\}$ and we can take the points $\{x_n\}$ to be those corresponding to their intervals in the partitions P_j.

Let ε be positive. There is a positive integer n such that $1/n < \varepsilon$, then there is an integer q such that for $j > q$, $I_j \subset [b-\delta_n(b),b)$ and finally there is an integer z such that for $k > z$ we have $J_k \subset I_j$ for some $j > q$. If now $r > s > z$, then $J_{s+1}, J_{s+2}, \ldots, J_r$ and $x_{s+1}, \ldots, x_r$ form a δ_n-fine partition P of an interval in $(b-\delta_n(b),b)$, so that $|\Sigma_{j=s+1}^{r} f(x_j)\alpha(J_j)| = |\sigma(f, P, \alpha)| < 1/n < \varepsilon$. By the Cauchy Criterion the series $\Sigma_{n=1}^{\infty} f(x_n)\alpha(J_n)$ converges.

Theorem 5. If the function f has Property Σ with respect to the gap function α then f has Property S with respect to α.

Proof. Let the gap function α be determined by α' and Q and suppose the function f has Property Σ with respect to α. Let $\{x_n\}$ be a monotone sequence in Q which we suppose to be increasing. Let K be the set of all x_n's together with the supremum of this set. Finally, let $\{s_k\}$ be an indexing of Q. There is a positive function δ on I with the following properties.

(1) $\delta(x)$ is less than the distance from x to $K - \{x\}$ for each x not the limit of $\{x_n\}$.

(2) $|f(s_k)|\alpha([s_k - \delta(s_k), s_k + \delta(s_k)]) < |f(s_k)|\alpha'(s_k) + 1/2^k$ for each s_k in Q.

For each n let I_n be an interval with midpoint x_n and satisfying $I_n \subset (x_n - \delta(x_n), x_n + \delta(x_n))$. By Property Σ there is a monotone sequence of intervals $\{J_k\}$ and points $\{y_k\}$ such that $y_k \in J_k$, $\{J_k; y_k\}$ is δ-fine, each I_n is a union of some J_k's and $\Sigma_{k=1}^{\infty} f(y_k)\alpha(J_k)$ converges. By property 1 of δ each x_n is in the set $\{y_k\}$. Since we will now be concerned only with convergence of certain series, we may as well assume that each y_k belongs to Q, since f has value 0 off Q.

Let ε be positive. There is a positive integer T such that $\Sigma_{i=T}^{\infty} 1/2^i < \varepsilon/2$ and $|\Sigma_{i=R}^{S} f(y_i)\alpha(J_i)| < \varepsilon/2$ whenever $T < R < S$. For such R and S we have $|\Sigma_{j=R}^{S} f(y_j)\alpha'(y_j)| \leq \Sigma_{j=R}^{S} |f(y_j)|(\alpha(J_j) - \alpha'(y_j)) + |\Sigma_{j=R}^{S} f(y_j)\alpha(J_j)| < \Sigma_{j=T}^{S} 1/2^j + \varepsilon/2 < \varepsilon$. By the Cauchy criterion the series $\Sigma_{j=1}^{\infty} f(y_j)\alpha'(y_j)$ converges so that f has Property S.

Theorem 6. If the function f has Property Σ with respect to α on the interval $[a, b]$ and f is P-integrable with respect to α on $[a, x]$ whenever $a < x < b$, then f is P-integrable with respect to α on $[a, b]$.

Proof. If α is not continuous at b, then set

$$\bar{\alpha}(x) = \begin{cases} \alpha(x) & \text{if } x < b \\ \lim\limits_{t \to b^-} \alpha(t) & \text{if } x = b \end{cases} \quad \text{and} \quad \hat{\alpha}(x) = \alpha(x) - \bar{\alpha}(x).$$

Then $\alpha = \bar{\alpha} + \hat{\alpha}$ and certainly f satisfies the hypotheses of the theorem with respect to $\bar{\alpha}$. Since f is integrable with respect to $\hat{\alpha}$, we will be done if it is integrable with respect to $\bar{\alpha}$, which is continuous at b.

We may thus suppose that α is continuous at b. For such an α there are no improper Perron integrals ([3, p. 37]), so we need only show that $\lim_{t \to b^-} P\int_a^t f d\alpha$ exists. To do this we need only show that for each positive ε there is a positive r such that $|P\int_s^t f d\alpha| < \varepsilon$ whenever $b - r < s < t < b$.

Suppose this last statement is false. Then there is a positive ε for which no positive r works. From this we get nonoverlapping intervals I_n which increase to b and have $|P\int_{I_n} f d\alpha| \geq \varepsilon$. On the interval I_n there is a positive function δ_n such that $|\sigma(f, P) - P\int_{I_n} f d\alpha| < \varepsilon/3$ whenever P is a δ_n-fine partition of I_n. Extend the δ_n's any way to a positive function δ. By Property Σ there is a monotone sequence of intervals $\{J_k\}$ with points $x_k \in J_k$ such that each I_n is a union of J_k's, $\{J_k; x_k\}$ is δ-fine and $\Sigma_{k=1}^{\infty} f(x_k)\alpha(J_k)$ converges. Then there are integers m, n and q so that $\{J_m, \ldots, J_n; x_m, \ldots, x_n\}$ is a partition P of I_q with $|\sigma(f, P, \alpha)| < \varepsilon/3$. Since P is a δ_q-fine partition of I_q we have $|\sigma(f, P, \alpha) - P\int_{I_q} f d\alpha| < \varepsilon/3$ so that $|P\int_{I_q} f d\alpha| < \varepsilon$, a contradiction which establishes the theorem.

Since Property Σ is a "two-sided" property there is clearly a theorem analogous to Theorem 6 for integrability at the left endpoint of an interval. Combining these two results yields the following.

Corollary 1. If the function f has Property Σ with respect to α on the interval I, then $\{x|f$ is not P-integrable with respect to α on any neighborhood of $x\}$ has no isolated points.

Theorem 7. Let α be the gap function determined by Q and α', where Q lies in the interior of the interval I and has countable closure. A function f on I is P-integrable with respect to α on I if and only if f has Property Σ with respect to α on I.

Proof. The necessity of Property Σ follows from Theorem 4. To verify sufficiency, note that the set K of points which have no neighborhood on which f is P-integrable must be a subset of the closure of Q and so must be countable. Further, K must be closed and by Corollary 1 K cannot have an isolated point. But the only compact countable set with no isolated point is empty. The result follows.

We note that if α is any nondecreasing function whose support is countable, then there is a set Q and a function α' for which α is the gap function determined by Q and α' and further that the closure of Q must be countable, so Theorem 7 applies to such functions α.

SOME COUNTEREXAMPLES. We first give the example we promised earlier which shows that Property S does not imply P-integrability if the set Q has infinitely many limit points.

Let $q_{n,i} = (1/n) + (1/(n(n-1) + i))$ for $n, i = 1,2, \ldots$. Let Q be the set of all $q_{n,i}$'s. Let $\alpha'(q_{n,1}) = 1/2^{n+i}$. Let $A = \Sigma_{n=1}^{\infty}(-1)^{n+1}/n$

and $f(q_{n,i}) = ((-1)^{i+1}2^{n+i})/(niA)$ while $f(x) = 0$ for other x's. Using Theorem 2 it is easy to see that $P\int_{1/n} f d\alpha = \Sigma_{k=1}^{n} 1/k$. Further, Lemma 1 easily implies that f is not P-integrable on $I = [0, 3]$ because of its behavior at 0.

Alas, f has Property S. Increasing sequences in Q must be finite and so cause no trouble. If a sequence decreases to limit a with $a \neq 0$, then the sequence lies in the interval $[a, 1]$ on which f is P-integrable and again there is no trouble by Theorems 4 and 5.

So suppose a sequence $\{x_n\}$ decreases with limit 0. First, take the largest i for which $q_{1,i}$ appears among the x_n's. Fill in the sequence with all $q_{1,j}$ for $j < i$, where by "fill in" we will mean "insert in the sequence so as to obtain another decreasing sequence". Then, if the resulting sum through $f(q_{1,i})\alpha'(q_{1,i})$ differs from 0 by more than 1, fill in with enough $q_{1,j}$'s with $f(q_{1,j})$ of the proper sign so that the resultant sum is within 1 of 0.

Now take the last $q_{2,i}$ appearing among the x_n's. Proceed as above so that we end with a partial sum within $1/2$ of 0. Continue in this way inductively through $\{q_{3,i}\}$, $\{q_{4,i}\}$, ... The resulting decreasing sequence $\{y_k\}$ is easily shown to satisfy $\Sigma_{k=1}^{\infty} f(y_k)\alpha'(y_k) = 0$ so that f has Property S.

We now present an example to show that Property Σ is not equivalent to P-integrability. Let $Q = \{t_i\}$ be the dyadic rationals in $(0, 1)$ indexed in any order. When p is an odd integer between 0 and 2^k we define $\alpha'(p/2^k) = 1/2^{2k-1}$ and $f(p/2^k) = (-2)^k/k$. As usual we let the values of α' and f be 0 at points not in Q. It is not hard to see that $f(t_i)\alpha'(t_i) \to 0$ as $i \to \infty$ and that neither $f^+ = (f + |f|)/2$ nor $f^- = (f - |f|)/2$ are integrable with respect to α on any subinterval of $I = [0, 1]$.

From [5] we know that a P-integrable function must be L-integrable on a neighborhood of each point of some dense set in I. But then both f^+ and f^- must be integrable on this neighborhood by properties of the Lebesgue integral. Consequently, the particular f of this example cannot be P-integrable.

Finally, we show that f has Property Σ with respect to α. Let $\{I_n\}$ be a monotone sequence of intervals and δ a positive function on I. We may assume that for each i $\delta(t_i)$ is so small that $|f(t_i)|(\alpha([t_i - \delta(t_i), t_i + \delta(t_i)]) - \alpha'(t_i)) < 1/2^i$.

We may suppose that there are points $a_0 < a_1 < \ldots$ for which $I_k = [a_{k-1}, a_k]$. For each k there is a δ-fine partition P_k of I_k such that the point corresponding to the interval containing a_{k-1} or a_k is a_{k-1} or a_k respectively and the point corresponding to any other interval lies in the interior of the interval. From these partitions we get a monotone sequence of intervals $J_i = [b_{i-1}, b_i]$ with points x_i such that $\{J_i, x_i\}$ is δ-fine and each I_k is a union of J_i's.

Suppose that $f(x_1) < 0$. The set $F^+ = \{x | f(x) \geq 0\}$ has a countable complement in I and so is an absolute G_δ-set. Thus using the Baire Category Theorem we can find a positive integer n and a small interval $K = [r_1, s']$ in J_1 of length less than $1/n$ so that $x_1 < r_1$, $x_2 - \delta(x_2) < r_1$

and $D = \{x \in F^+ | \delta(x) > 1/n\} \cap K$ is dense in K.

Since $f(t_i)\alpha'(t_i) \to 0$ as $i \to \infty$, the set $E = \{t_i | f(t_i)\alpha'(t_i) \geq 1\}$ is finite. Since f^+ is not integrable on K the sum $\Sigma\{f^+(t_i)\alpha'(t_i) | t_i \in K-E\}$ does not converge, so there is a finite set $A \subset Q \cap K - E$ for which $\Sigma\{f(t_i)\alpha'(t_i) | t_i \in A\} + f(x_1)\alpha([b_0, r_1]) > 0$. Each point t of A is contained in the interior of a small interval J_t such that the J_t's are pairwise disjoint and $f(t)\alpha(J_t) < 1$ for each $t \in A$.

Now $D - E$ is dense in K so each complementary interval of $\cup\{J_t | t \in A\}$ in K contains a point of $D - E$. These complementary intervals with corresponding points in $D - E$ together with each J_t with corresponding point t form a δ-fine partition P' of K such that each term of $\sigma(f, P')$ is nonnegative but less than 1 and $\sigma(f, P') + f(x_1)\alpha([b_0, r_1]) > 0$.

Beginning from r_1, take as many intervals of P' as necessary to get a δ-fine Perron partition P'' of an interval $[r_1, s_1]$ so that $|\sigma(f, P'') + f(x_1)\alpha([b_0, r_1])| < 1$. Let $J_2' = [s_1, b_2]$ and let P_1 be P'' together with $[b_0, r_1]$ with point x_1. If $f(x_1) > 0$ then we can carry out exactly the same process using F^- and f^- in place of F^+ and f^+.

Suppose that $b_2 \neq a_1$. The same technique delivers a small interval $[r_2, s_2]$ in J_2' with a δ-fine partition P' such that $x_2 < r_2$, $x_3 - \delta(x_3) < r_2$, each term of $\sigma(f, P')$ has absolute value less than 1/2 and sign opposite that of $\sigma(f, P_1) + f(x_2)\alpha([s_1, r_2])$ and if P_2 is composed of P_1, P' and $[s_1, r_2]$ with point x_2 then $|\sigma(f, P_2)| < 1/2$.

Suppose $b_z = a_1$. We can continue as above until we obtain a δ-fine partition P_{z-1} of an interval $[a_0, s_{z-1}]$. Let P_z be composed of P_{z-1} and the interval $[s_{z-1}, b_z]$ with point $x_z = b_z = a_1$. Now continue as before, treating the interval J_{z+1} as we treated J_1 to obtain a δ-fine partition P_{z+1} of an interval $[a_0, s_{z+1}]$ having properties analogous to those of $P_1, P_2, \ldots, P_{z-1}$, for example, $|\sigma(f, P_{z+1})| < 1/(z + 1)$.

Continuing in this way we obtain a monotone sequence of intervals $\{K_i\}$ and points $\{y_i\}$ such that $\{K_i; y_i\}$ is δ-fine, $f(y_i)\alpha(K_i) \to 0$ as $i \to \infty$, each I_q is a union of K_i's and there is an increasing sequence of integers $\{k_j\}$ such that $|\Sigma_{i=1}^{z} f(y_i)\alpha(K_i)| < 1/(j - 1)$ whenever z is an integer with $k_{j-1} < z < k_j$ and $j = 2, 3, \ldots$.

It is not difficult to use these facts to show that $\Sigma_{i=1}^{\infty} f(y_i)\alpha(K_i) = 0$. Hence f has Property Σ with respect to α on I.

REFERENCES

[1] McLeod, R.M., The Generalized Riemann Integral, Carus Mathematical Monograph # 20, 1980, American Mathematical Association, Washington, D.C.

[2] Natanson, I.P., Theory of Functions of a Real Variable, Vol. II, translated by L.F. Boron, Frederick Ungar Publishing Co., New York, 1960.

[3] Pfeffer, W.F., The Riemann-Stieltjes Approach to Integration, Technical Report 187, National Research Institute for Mathematical Sciences, Pretoria, South Africa.

[4] Saks, S., Theory of the Integral (2nd revised edition), translated by L.C. Young, Dover Publications, New York, 1964.

[5] Schurle, A.W., Perron Integrability Versus Lebesgue Integrability, UPM Department of Mathematical Sciences Technical Report No. 60, 1983.

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