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UNIQUENESS OF DERIVATIVES OF FUNCTIONS DEFINED ON CLOSED SETS.

0. Introduction.

In [6], Whitney introduced the space $C^k(F)$ of k times differentiable functions on an arbitrary closed subset F of $\mathbb{R}^n$, and showed that $C^k(F)$ is the trace on F of the space $C^k(\mathbb{R}^n)$. The elements in $C^k(F)$ are families $\{f^{(j)}\}_{|j|\leq k}$, where one can think of the functions $f^{(j)}$ as derivatives of $f^{(0)}$. In general, the functions $f^{(j)}$ are not uniquely determined by $f^{(0)}$, and so it is natural to ask for which sets F one has uniqueness. This problem was solved by Glaeser in [1], Chapter III, §5 (see also [2], Chapter II, §3). Here we consider the corresponding problem for the Lipschitz spaces $\mathrm{Lip}(\alpha,F)$ and $\Lambda_\alpha(F)$. The condition for uniqueness of derivatives in these spaces depends on α; in this respect the situation is not the same as in the $C^k(F)$-case, where the condition is independent of k. The main result is given in Theorem 2 in §3. As a tool, we first prove in §2 a theorem on a general kind of multiplication of functions in the Lipschitz classes; this theorem also has a counterpart in Glaeser [1].

The following notation will be used throughout the paper. F denotes a closed subset of the Euclidian space $\mathbb{R}^n$ with points $x=(x_1,x_2,\ldots,x_n)$, α a positive real number, and k the integer such that $k<\alpha\leq k+1$. The letter c denotes a constant, not necessarily the same each time it appears. $B(x,r)$ is a closed ball with center x and radius r. We use the usual multiindex notation: If $j=(j_1,j_2,\ldots,j_n)$ is a multiinteger of length $|j|=j_1+j_2+\ldots+j_n$, then D^jf is the corresponding partial derivative of f of order $|j|$, $x^j=x_1^{j_1}\ldots x_n^{j_n}$ and $j!=j_1!j_2!\ldots j_n!$.

1. The spaces $\mathrm{Lip}(\alpha,F)$ and $\Lambda_\alpha(F)$.

The classical definition of these spaces in the case when $F=\mathbb{R}^n$ is as follows: A function f defined on $\mathbb{R}^n$ belongs to the Lipschitz space $\Lambda_\alpha(\mathbb{R}^n)$ if and only if f is k times continuously differentiable and there exists a constant M such that $|D^j f|\leq M$, $|j|\leq k$, and the derivatives $D^j f$ of order $|j|=k$ satisfy $|D^j f(x+h)-2D^j f(x)+D^j f(x-h)| \leq M|h|^{\alpha-k}$. Taking instead the first difference, we obtain the space $\mathrm{Lip}(\alpha,\mathbb{R}^n)$. If α is noninteger, the two spaces coincide.

Lipschitz spaces have also been defined on arbitrary closed sets F. The spaces $\mathrm{Lip}(\alpha,F)$ are studied e.g. in [5], while the spaces $\Lambda_\alpha(F)$ were introduced in [2]. An extensive treatment of these spaces is given in [3]. We shall now list a number of their properties which we need, referring to [3] for details. The main result for the spaces $\Lambda_\alpha(F)$ is a trace theorem, stating that the pointwise restriction to F of the derivatives $\{D^j f\}$, $|j|\leq k$, of a function in $\Lambda_\alpha(\mathbb{R}^n)$ is an element in $\Lambda_\alpha(F)$, and conversely every element in $\Lambda_\alpha(F)$ may be extended to a function defined on $\mathbb{R}^n$ belonging to $\Lambda_\alpha(\mathbb{R}^n)$. The corresponding result for $\mathrm{Lip}(\alpha,F)$ is a version of the classical Whitney extension theorem. There are many equivalent definitions of the space $\Lambda_\alpha(F)$, and we choose here one suitable for our purpose, see [3], Chapter III, §3, Proposition 4. The functions $f^{(j)}$ in the definition may be thought of as derivatives of $f^{(0)}$.

DEFINITION 1. Let N be an integer, $N\geq[\alpha]$, and $\{f^{(j)}\}_{|j|\leq k}$ a family of functions defined on F. Then $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$ if and only if the following conditions hold:

For every cube Q with sides of length $\delta\leq 1$ and intersecting F there exists a polynomial P_Q of degree at most N so that

a) $|f^{(j)}(x)-D^jP_Q(x)| \leq M\delta^{\alpha-|j|}$, $|j|\leq k$, $x\in Q\cap F$

b) $|P_Q(x)| \leq M$, $x\in Q$, if $\delta=1$, and

c) if $Q'\subset Q$ is a cube intersecting F and having side δ' such that $\delta\leq 2\delta'$, then

$$|P_Q(x)-P_{Q'}(x)| \leq M\delta^{\alpha}, \quad x\in Q'.$$

The norm of $\{f^{(j)}\}_{|j|\leq k} \in \Lambda_\alpha(F)$ is the infimum of the possible constants M.

Different values of the integer N above give rise to equivalent spaces. The space $\mathrm{Lip}(\alpha,F)$ is obtained if we take polynomials of degree $\leq k$ instead of degree $\leq N$ in the definition above; in this case condition c) may be omitted. The spaces $\mathrm{Lip}(\alpha,F)$ and $\Lambda_\alpha(F)$ coincide if α is non-integer. If Q has sidelength $\delta\leq 1$ and P_Q is associated to a family $\{f^{(j)}\}_{|j|\leq k}$ belonging to $\Lambda_\alpha(F)$ or $\mathrm{Lip}(\alpha,F)$ as in the definition, then $|D^jP_Q(x)|\leq cM$, $x\in Q$, for $|j|\leq k$, where the constant c only depends on k and n, and if $|j|=k+1=\alpha$ we have in the Λ_α-case $|D^jP_Q(x)| \leq cM(1+|\ln\delta|)$, $x\in Q$. This follows from Lemma 1, Lemma 2 and Remark 4 in Chapter III of [3]. Taking Q with $\delta=1$, we get $|f^{(j)}| \leq cM$, $|j|\leq k$.

The definition of $\Lambda_\alpha(F)$ may be simplified if we consider more special sets F. Suppose that F preserves Markov's inequality in the sense that for all positive integers K, for all polynomials P of degree at most K and all balls $B=B(x_0,r)$ with $x_0\in F$ and $0<r\leq 1$ we have $\max|\mathrm{grad}\, P| \leq c(F,n,K)\max r^{-1}|P|$, where the maximum is taken over $F\cap B$. Then the functions $f^{(j)}$ are uniquely determined by $f^{(0)}$ if the family $\{f^{(j)}\}_{|j|\leq k}$ belongs to $\Lambda_\alpha(F)$ or $\mathrm{Lip}(\alpha,F)$, and we may identify the family with the single function $f=f^{(0)}$. Furthermore, $f\in\Lambda_\alpha(F)$ if and only if $|f|\leq M$ and for every cube Q with sides of length $\delta\leq 1$ (the number 1 may of course be replaced by any number $\delta_0>0$) and with center in F there is a polynomial

P_Q of degree at most $[\alpha]$ such that $|f-P_Q|\leq M\delta^\alpha$ on $Q\cap F$. Replacing $[\alpha]$ by k we get a characterization of $Lip(\alpha,F)$. Examples of sets preserving Markov's inequality are closed balls and the whole space. If $F=\mathbb{R}^n$, then the spaces $\Lambda_\alpha(F)$ and $Lip(\alpha,F)$ are equivalent to the classical spaces $\Lambda_\alpha(\mathbb{R}^n)$ and $Lip(\alpha,\mathbb{R}^n)$ as defined in the beginning of this section. Also for certain nice subsets of $\mathbb{R}^n$, the Lipschitz spaces may be defined by means of differences. Consider for example the case when F is a closed ball B with interior B_0. Then $f\in\Lambda_\alpha(F)$ if and only if f is continuous on B and $f\in\Lambda_\alpha(B_0)$ in the sense that f is k times continuously differentiable in B_0 with bounded derivatives and, for $|j|=k$ and $x,x-h,x+h\in B_0$, $|D^jf(x+h)-2D^jf(x)+D^jf(x-h)|\leq c|h|^{\alpha-k}$. The equivalence of this definition and the one given above for sets preserving Markov's inequality, follows e.g. from the fact that the trace theorem has been proved in both cases, see [4], pp. 380-383, cf. also [2], §4.1.

It is well-known that $\Lambda_\alpha(\mathbb{R}^n)$ and $Lip(\alpha,\mathbb{R}^n)$ are algebras, and using the trace theorem we see that $\Lambda_\alpha(F)$ and $Lip(\alpha,F)$ are algebras if F preserves Markov's inequality. We will also need the following result. The corresponding lemma for $Lip(\alpha,\mathbb{R}^n)$ is also valid; to see this replace $[\alpha]$ by k in the proof.

LEMMA 1. <u>Suppose that</u> $f\in\Lambda_\alpha(\mathbb{R}^n)$ <u>and that</u> $f(x_0)\neq 0$. <u>Then there exists a closed ball</u> $B=B(x_0,r)$ <u>such that</u> $1/f\in\Lambda_\alpha(B)$.

<u>Proof</u>. Let B and s be so that $|f|\geq s>0$ on B, and let Q be a cube centered in F with sidelength δ satisfying $0<\delta\leq\delta_0$, where δ_0 is a fixed number chosen so small that $M_0\delta_0^\alpha < s/4$, where M_0 is the $\Lambda_\alpha(F)$-norm of f. Choose a polynomial P_Q as in Definition 1 with $N=[\alpha]$ and the constant M less than $2M_0$. Then we have $|f-P_Q|\leq M\delta^\alpha < 2M_0\delta_0^\alpha < s/2$ on $B\cap Q$, so $|P_Q|\geq s/2$ on $B\cap Q$, and thus $|1/f-1/P_Q|=|(P_Q-f)/(fP_Q)|\leq c\delta^\alpha/s^2=c\delta^\alpha$ on $B\cap Q$.

We shall prove that the inequality $|1/f-1/P_Q|\leq c\delta^{\alpha}$ remains true on $B\cap Q$ if we replace $1/P_Q$ by some polynomial R_Q of degree $\leq[\alpha]$. By the characterization of $\Lambda_\alpha(F)$ when F preserves Markov's inequality it then follows that $1/f\in\Lambda_\alpha(B)$, since $|1/f|\leq 1/s=c$ on B.

Let R_Q be the Taylor polynomial of order $[\alpha]$ of $1/P_Q$ expanded about the center of Q. Then $|1/P_Q-R_Q|\leq c\delta^{[\alpha]+1}\sup|D^j(1/P_Q)(\xi)|$ on Q, where the supremum is taken over all j with $|j|=[\alpha]+1$ and all $\xi\in Q$. Since $D^jP_Q=0$ if $|j|=[\alpha]+1$, $D^j(1/P_Q)$ is a sum, where each term is a product of factors $D^{\ell}P_Q$, $|\ell|\leq[\alpha]$ and factors $1/P_Q$, the number of factors being less than a constant, say m, depending on α. By the estimates on $D^{\ell}P_Q$ given after Definition 1, such a term is $\leq c$ if no multi-index ℓ is of order $[\alpha]=k+1$, and if some are, then at least the term is less than $c(1+|\ln\delta|)^m$. This gives $|1/P_Q-R_Q|\leq c\delta^{[\alpha]+1}(1+|\ln\delta|)^m\leq c\delta^{\alpha}$ on Q since $\alpha<[\alpha]+1$, and thus $|1/f-R_Q|\leq c\delta^{\alpha}$ on $B\cap Q$.

2. The product of functions in Lipschitz spaces.

If $f\in\Lambda_\alpha(\mathbb{R}^n)$ and $g\in\Lambda_\beta(\mathbb{R}^n)$, then $fg\in\Lambda_\gamma(\mathbb{R}^n)$, where $\gamma=\min(\alpha,\beta)$, but in general fg does not belong to $\Lambda_{\gamma'}(\mathbb{R}^n)$ for some $\gamma'>\gamma$. On subsets of $\mathbb{R}^n$, the situation is more varied, if one defines the "product" of families of functions in a suitable way. For the spaces $C^k(F)$ this was shown by Glaeser in [1], p.33-34. In this section k_α will denote the nonnegative integer such that $k_\alpha<\alpha\leq k_\alpha+1$, and analogously for β and γ. In the statement of the theorem below the following definition is needed.

DEFINITION 2. Let ℓ be a nonnegative integer, $\ell<\alpha$. A family $f=\{f^{(j)}\}_{|j|\leq k_\alpha}$ defined on F is of <u>type</u> (ℓ,α) if $f\in\Lambda_\alpha(F)$ and $f^{(j)}=0$ on F for $|j|<\ell$.

Thus, any function in $\Lambda_\alpha(F)$ is of type $(0,\alpha)$. When we below say that we "formally compute the jth derivative of $f^{(0)}g^{(0)}$", we consider the functions $f^{(j)}$ as derivatives of $f^{(0)}$ and differentiate as if $f^{(0)}$ and $g^{(0)}$ were both j times differentiable. Thus $h^{(j)}$ in the theorem is given by $\sum_\nu \frac{j!}{\nu!(j-\nu)!} f^{(\nu)}g^{(j-\nu)}$, where the sum is taken over all multiindices $\nu\leq j$ such that $f^{(\nu)}$ and $g^{(j-\nu)}$ are both defined. In the theorem, all functions are defined on a closed set F.

THEOREM 1. <u>Suppose that</u> f <u>is of type</u> (ℓ,α) <u>and</u> g <u>of type</u> (m,β), <u>where</u> $0\leq\ell<\alpha$ <u>and</u> $0\leq m<\beta$ <u>and let</u> $\gamma=\min(\alpha+m,\beta+\ell)$. <u>Then</u> $h=\{h^{(j)}\}_{|j|\leq k_\gamma}$ <u>is of type</u> $(\ell+m,\gamma)$, <u>where</u> $h^{(j)}$ <u>is the function obtained by formally computing the jth derivative of</u> $f^{(0)}g^{(0)}$ <u>(see above) by means of Leibnitz´ rule, interpreting a product</u> $f^{(\nu)}g^{(\mu)}$, $j=\nu+\mu$, <u>as zero if one of the factors is not defined.</u>

Note that if $|j|\leq k_\gamma$, $j=\nu+\mu$, and, say, $|\nu|\geq k_\beta+1$ so that $f^{(\nu)}$ is not defined, then $|\mu|=|j|-|\nu| < \gamma-\beta \leq \beta+\ell-\beta = \ell$, so $g^{(\mu)}=0$. Thus in the interpretation of the product $f^{(\nu)}g^{(\mu)}$ in the theorem, we adopt the convention that a product of a derivative which is not defined and one which is zero, is zero.

<u>Proof of Theorem 1.</u> It is clear that $h^{(j)}=0$ if $|j|<\ell+m$, and we shall prove that $h\in\Lambda_\gamma(F)$. For every cube Q with sidelength $\delta\leq 1$, intersecting F, choose, as in Definition 1, polynomials $P_{f,Q}=P_f$ and $P_{g,Q}=P_g$ of degrees $\leq[\alpha]$ and $\leq[\beta]$ respectively, so that $|f^{(j)}-D^jP_f|\leq c\delta^{\alpha-|j|}$ on $F\cap Q$, $|j|\leq k_\alpha$, $|P_{f,Q}-P_{f,Q'}|\leq c\delta^\alpha$ on $Q'\subset Q$ if $\delta\leq 2\delta'$, $|P_f|\leq c$ on Q if $\delta=1$, and similarily for P_g. Define P_Q by $P_Q=P_fP_g$; then P_Q is a polynomial of degree at most $N=[\alpha]+[\beta]$. According to Definition 1 to prove that $h\in\Lambda_\gamma(F)$ we shall show that

a) $|h^{(j)}-D^jP_Q| \le c\delta^{\gamma-|j|}$ on $Q\cap F$, $|j|\le k_\gamma$,

b) $|P_Q-P_{Q'}| \le c\delta^{\gamma}$ on $Q'\subset Q$ if $\delta\le 2\delta'$, and

c) $|P_Q|\le c$ on Q if $\delta=1$.

Because of the definition of $h^{(j)}$ and P_Q, a) follows if we prove that on $F\cap Q$, for $|j|\le k_\gamma$, $j=\nu+\mu$,

$$|f^{(\nu)}g^{(\mu)}-D^\nu P_f D^\mu P_g| \le c\delta^{\gamma-|j|} \quad \text{if} \quad |\nu|\le k_\alpha \quad \text{and} \quad |\mu|\le k_\beta \tag{1}$$

and

$$|D^\nu P_f D^\mu P_g| \le c\delta^{\gamma-|j|} \quad \text{if} \quad |\nu|>k_\alpha \quad \text{or} \quad |\mu|>k_\beta. \tag{2}$$

The left side of the inequality (1) is less than $|f^{(\nu)}-D^\nu P_f||g^{(\mu)}|+|D^\nu P_f||g^{(\mu)}-D^\mu P_g| = S_1+S_2$. If $|\mu|<m$, then $g^{(\mu)}=0$ since g is of type (m,β), and if $|\mu|\ge m$, then, as we saw after Definition 1, $|g^{(\mu)}|\le c$, so $S_1\le c\delta^{\alpha-|\nu|}=c\delta^{\alpha+|\mu|-|j|} \le c\delta^{\alpha+m-|j|} \le c\delta^{\gamma-|j|}$. Similarily, if $|\nu|\ge \ell$, then $S_2\le c\delta^{\beta-|\mu|} \le c\delta^{\beta+\ell-|j|} \le c\delta^{\gamma-|j|}$, and if $|\nu|<\ell$, then $S_2\le c\delta^{\alpha-|\nu|}\delta^{\beta-|\mu|}=c\delta^{\alpha+\beta-|j|}\le c\delta^{\gamma-|j|}$. Thus we always have $S_1+S_2\le c\delta^{\gamma-|j|}$ which proves (1).

By symmetry, it is enough to consider the case $|\nu|>k_\alpha$ in the proof of (2). If α is noninteger, then $D^\nu P_f=0$ if $|\nu|>k_\alpha=[\alpha]$ since P_f has degree $\le[\alpha]$, so we assume that $|\nu|=k_\alpha+1=\alpha$. Then (see the remark following Definition 1) $|D^\nu P_f| \le c(1+|\ln\delta|)$, and we have $|\mu|=|j|-|\nu|<\gamma-\alpha\le\alpha+m-\alpha$, so $|D^\nu P_f D^\mu P_g|=|D^\nu P_f||D^\mu P_g-g^{(\mu)}| \le c(1+|\ln\delta|)\delta^{\beta-|\mu|} = c(1+|\ln\delta|)\delta^{\beta+\alpha-|j|}\le c\delta^{\gamma-|j|}$ since $\gamma<\alpha+\beta$, and (2) is proved.

Next we prove b), putting $P_{f,Q'}=P_f'$. Then

$|P_Q-P_{Q'}|=|P_fP_g-P_f'P_g'|\le|P_f-P_f'||P_g|+|P_f'||P_g-P_g'|=T_1+T_2$.

To estimate $|P_g|$, let x_0 be the center of Q', and write $P_g(x)=\sum_{|\eta|\le[\beta]} A_\eta(x)$, where $A_\eta(x)=D^\eta P_g(x_0)(x-x_0)^\eta/\eta!$. Let now $x\in Q'$. If $|\eta|<m$, then $|A_\eta|\le c\delta^{\beta-|\eta|}\delta^{|\eta|}=c\delta^{\beta}\le c\delta^{m}$, if $m\le|\eta|\le k_\beta$, then $|A_\eta|\le c\delta^{|\eta|}\le c\delta^{m}$,

and if $|\eta|=k_\beta+1$ and $\beta=k_\beta+1$ then $|A_\eta|\le c(1+|\ln\delta|)\delta^\beta\le c\delta^m$; here we used the estimates on derivatives given after Definition 1. Thus we have $T_1\le c\delta^\alpha\delta^m\le c\delta^\gamma$. Since the term T_2 is handled in the same way, b) is proved. Since c) is immediate, the proof of the theorem is finished.

The result above holds if we replace $\Lambda_\alpha(F)$ by $\mathrm{Lip}(\alpha,F)$ in Definition 2 and Theorem 1. According to the definition of $\mathrm{Lip}(\gamma,F)$ we must prove that the statements we called a) and c) in the proof above hold with a polynomial P_Q of degree $\le k_\gamma$ instead of $\le N$, $N\ge[\gamma]$. Taking $P_Q=P_fP_g$ as above (with the degree of $P_f\le k_\alpha$), one sees that a) and c) hold with an analogous, but simpler, argument as in the Λ_α-case. We must replace P_Q with a polynomial L_Q of degree $\le k_\gamma$. Let x_0 be the center of Q, and put $L_Q=P_Q-R_Q$, where $R_Q(x)=\sum_{k_\gamma<|\eta|\le N} D^\eta P_Q(x_0)(x-x_0)^\eta/\eta!$, $N=k_\alpha+k_\beta$. Since $D^\eta P_Q$ is a sum of terms of the form $D^\nu P_f D^\mu P_g$, $\eta=\nu+\mu$, and since all derivatives of P_f and P_g are bounded on Q, we see that $|R_Q|\le c\delta^{k_\gamma+1}\le c\delta^\gamma$ on Q and, by Markov's inequality, $|D^jR_Q|\le c\delta^{\gamma-|j|}$ on Q. This shows that a) and c) hold with P_Q replaced by L_Q, a polynomial of degree $\le k_\gamma$.

3. Uniqueness of derivatives.

We shall now give a general theorem, which in particular characterizes the sets F, for which the functions $f^{(j)}$ are uniquely determined by $f^{(0)}$ if $\{f^{(j)}\}_{|j|\le k}\in\Lambda_\alpha(F)$. Analogous results are valid for $\mathrm{Lip}(\alpha,F)$: Replace $\Lambda_s(\mathbb{R}^{n-1})$ by $\mathrm{Lip}(s,\mathbb{R}^{n-1})$ in Definition 3 and $\Lambda_\alpha(F)$ by $\mathrm{Lip}(\alpha,F)$ in Theorem 2. The proof is the same. To simplify the formulation of the theorem, we make the following definition.

DEFINITION 3. F has the property $G(s)$, $s>1$, if and only if the following does not hold: There is a point $x_0 \in F$, an open neighbourhood U of x_0 and an $(n-1)$-dimensional Λ_s-surface M such that $F\cap U \subset M \cap U$.

Here, by an $(n-1)$-dimensional Λ_s-surface we mean a surface which for some i is of the form $\{x\in\mathbb{R}^n : x_i=\Phi(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n),\ \Phi\in\Lambda_s(\mathbb{R}^{n-1})\}$. Informally, the condition $G(s)$ means that no portion of F is contained in a Λ_s-surface.

Taking $m=k$ in the theorem below, we get the criteria for uniqueness of derivatives in $\Lambda_\alpha(F)$. An example of another consequence of the theorem is the following. If $f^{(0)}$ is as in the theorem, $\alpha>2$, and F has property $G(\alpha)$, then the functions $f^{(j)}$, $|j|=1$, are uniquely determined by $f^{(0)}$, but not all functions $f^{(j)}$, $|j|=2$, unless F has property $G(\alpha-1)$. As usual, $k<\alpha\leq k+1$, and the functions $f^{(j)}$ are all defined on F.

THEOREM 2. Let $\alpha>1$, let $f^{(0)}$ be such that there exist functions $f^{(j)}$ so that $\{f^{(j)}\}_{|j|\leq k} \in \Lambda_\alpha(F)$, and let $0<m\leq k$. Then the functions $f^{(j)}$, $|j|\leq m$, are uniquely determined by $f^{(0)}$ if and only if F has property $G(\alpha-m+1)$.

Before proving the theorem, we illustrate it by giving a corollary. The corollary follows using the trace theorem mentioned in the beginning of Section 1. Conversely, Theorem 2 in the case when $m=k$ follows from the corollary and the trace theorem.

COROLLARY 1. Let $\alpha>1$, $k<\alpha\leq k+1$, and consider the statements (A): $f\in\Lambda_\alpha(\mathbb{R}^n)$, $f(x)=0$, $x\in F$, and (B): $f\in\Lambda_\alpha(\mathbb{R}^n)$, $D^j f(x)=0$, $x\in F$, $|j|\leq k$. Then (A)$\Rightarrow$(B) if and only if F has property $G(\alpha-k+1)$.

Proof of Corollary 1. Assume first that F has property $G(\alpha-k+1)$ and that (A) holds. By the trace theorem, $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$, where $f^{(j)}=D^j f|F$.

Here $D^j f|F$ denotes the pointwise restriction of $D^j f$ to F. By Theorem 2, $f^{(j)}=0$, $|j|\leq k$, and hence (B) holds. Assume next that (A)⇒(B), let $\{f^{(j)}\}_{|j|\leq k}$ be any collection in $\Lambda_\alpha(F)$ and assume first that $f^{(0)}=0$. By the extension part of the trace theorem, there is a function $f\in\Lambda_\alpha(\mathbb{R}^n)$ such that $D^j f|F=f^{(j)}$. By our assumption it follows that $f^{(j)}=0$, $|j|\leq k$. Thus the functions $f^{(j)}$ are uniquely determined by $f^{(0)}$ if $f^{(0)}=0$, and hence in the general case also. Thus, by Theorem 2, F has property $G(\alpha-k+1)$.

<u>Proof of Theorem 2.</u> It is enough to consider the case when $f^{(0)}=0$. We first prove that if $f^{(j)}$, $|j|\leq m$, are not uniquely determined by $f^{(0)}$, then F has not property $G(\alpha-m+1)$. Let μ be a multiindex of least order such that $f^{(\mu)}$ is not identically zero, and take $x_0\in F$ such that $f^{(\mu)}(x_0)\neq 0$. By the extension theorem for $\Lambda_\alpha(F)$ (cf. the beginning of §1), there is a function $\mathcal{E}f\in\Lambda_\alpha(\mathbb{R}^n)$ such that $D^j(\mathcal{E}f)=f^{(j)}$ on F, $|j|\leq k$. Take ℓ of order $|\mu|-1$ so that, for some variable x_i, $D^\mu=\frac{\partial}{\partial x_i}D^\ell$.
Put $E=\{x:D^\ell(\mathcal{E}f)(x)=0\}$; then $F\subset E$. The version of the implicit function theorem given in Proposition 1 below, used with $g=D^\ell(\mathcal{E}f)\in\Lambda_{\alpha-|\mu|+1}(\mathbb{R}^n)$, gives us a certain neighbourhood U of x_0 of the form $U=\{x=(z,x_i):$ $z=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in\omega, x_i\in I\}$ and a function $\Phi\in\Lambda_{\alpha-|\mu|+1}(\bar{\omega})$ such that $E\cap U=\{x:x_i=\Phi(z), z\in\omega\}$. The function Φ may be extended to a function $\mathcal{E}_1\Phi\in\Lambda_{\alpha-|\mu|+1}(\mathbb{R}^n)\subset\Lambda_{\alpha-m+1}(\mathbb{R}^n)$, and thus $F\cap U\subset E\cap U=M\cap U$, where $M=\{x:x_i=\mathcal{E}_1\Phi(z),$ $z\in\mathbb{R}^{n-1}\}$ is an $(n-1)$-dimensional $\Lambda_{\alpha-m+1}$-surface. This proves one half of the theorem.

The proof of the other part of the theorem is obtained by means of Theorem 1. Assume that F has not property $G(\alpha-m+1)$, and take x_0, U, U bounded, and M, as in Definition 3; we may assume that M is given by $y=x_n=\Phi(z)$, $z=(x_1,x_2,\dots,x_{n-1})$, where $\Phi\in\Lambda_{\alpha-m+1}(\mathbb{R}^{n-1})$. Take $\psi\in C^\infty(\mathbb{R}^n)$ so that ψ is identically 1 in a neighbourhood of x_0 and zero outside U, and consider $L(z,y)=\psi(z,y)(y-\varphi(z))$. Then $L\in\Lambda_{\alpha-m+1}(\mathbb{R}^n)$ (both factors are, if we replace, as we can without changing L, the term y in the second factor by a C^∞-function of y with compact support which equals y on an interval big enough). Defining $\ell^{(j)}$ by $\ell^{(j)}(x)=D^jL(x)$, $x\in F$, the family $\ell=\{\ell^{(j)}\}_{|j|\leq k-m+1}$ belongs to $\Lambda_{\alpha-m+1}(F)$, and since $L=0$ on F, we have that ℓ is of type $(1,\alpha-m+1)$ (see Definition 2 in §2). If $m=1$ we are done, otherwise take f and g in Theorem 1 equal to ℓ, and construct a function $h=h_1$ of type $(2,\alpha-m+2)$ as in Theorem 1; then $h^{(j)}(x_0)=2!\neq 0$ if $j=(0,0,\dots,0,2)$. Next we take $f=h_1$ and $g=\ell$ in Theorem 1, and obtain a function h_2 of type $(3,\alpha-m+3)$ such that $h_2^{(j)}=3!$ if $j=(0,0,\dots,3)$. (By construction, $h_2^{(j)}$ is a formal sum of terms of the form $h_1^{(\nu)}\ell^{(\mu)}$ with $j=\nu+\mu$, where both factors are well defined and nonzero only if $|\nu|=2$; the coefficient of this term is 3.) Proceeding in this way, we obtain in the $(m-1)$th step a function h_{m-1} of type (m,α) such that $h_{m-1}^{(j)}=m!\neq 0$ if $j=(0,0,\dots,m)$. This concludes the proof of the theorem.

Finally we prove the version of the implicit function theorem used in the proof of Theorem 2. In the proposition, y denotes one of the variables x_i and z the other ones, $z=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$, and we write $x=(z,y)$ and $x_0=(z_0,y_0)$.

PROPOSITION 1. <u>Let</u> $\alpha>1$ <u>and</u> $g\in\Lambda_\alpha(\mathbb{R}^n)$, <u>and suppose that</u> $\partial g/\partial y(x_0)\neq 0$. <u>Then there is a neighbourhood</u> U <u>of</u> x_0 <u>of the form</u> $\{x=(z,y):z\in\omega,y\in I\}$, <u>where</u> ω <u>is an open ball in</u> $\mathbb{R}^{n-1}$ <u>with center</u> z_0 <u>and</u> I <u>an open interval with center</u> y_0, <u>and a function</u> $\Phi\in\Lambda_\alpha(\overline{\omega})$ <u>such that</u> $\{x:g(x)=0\}\cap U=\{x:y=\Phi(z), z\in\omega\}$.

<u>Proof</u>: We may assume that $i=n$, so $y=x_n$ and $z=(x_1,\dots,x_{n-1})$. By the im-

plicit function theorem we find U as in the proposition and a function Φ in $C^1(\omega)$, continuous in $\overline{\omega}$, satisfying $\frac{\partial\Phi}{\partial x_i}(z) = -\frac{\partial g}{\partial x_i}(z,\Phi(z))/\frac{\partial g}{\partial y}(z,\Phi(z))$, $z\in\omega$, $i=1,2,\ldots,n-1$.

We shall see that things may be arranged so that Φ actually belongs to $\Lambda_\alpha(\overline{\omega})$.

Since $\partial g/\partial y(x_0)\neq 0$ and $\partial g/\partial y\in\Lambda_{\alpha-1}(\mathbb{R}^n)$ there is by Lemma 1 a closed ball $B=B(x_0,r)$ such that $1/(\partial g/\partial y)\in\Lambda_{\alpha-1}(B)$. We may assume that B contains U. Since $\partial g/\partial x_i$ also belongs to $\Lambda_{\alpha-1}(B)$ and $\Lambda_{\alpha-1}(B)$ is an algebra, we have $\partial\Phi/\partial x_i(z)=H_i(z,\Phi(z))$, $z\in\omega$, for some $H_i\in\Lambda_{\alpha-1}(B)$. Then $H_i\in C^{k-1}(B_0)$ with bounded derivatives, where B_0 denotes the interior of B. Thus $\Phi\in C^k(\omega)$, with a derivative of order $\leq k$ given by a certain sum, where each term is a product of factors of the forms $D^\eta H_i(z,\Phi(z))$ and $D^\nu\Phi(z)$, where η and ν are multiindices of orders $\leq k-1$ and lengths n and $n-1$, respectively. To prove that $\Phi\in\Lambda_\alpha(\overline{\omega})$, it is (see §1) enough to show that the derivatives of orders $\leq k$ are bounded (this is obvious), and that the derivatives of order k belong to $\Lambda_{\alpha-k}(\omega)$, with $\Lambda_{\alpha-k}(\omega)$ defined as in §1 with differences. Since $D^\nu\Phi\in C^1(\omega)$ with bounded derivatives if $|\nu|\leq k-1$ we have (e.g. by means of the mean value theorem), since $0<\alpha-k\leq 1$, that $D^\nu\Phi\in\Lambda_{\alpha-k}(\omega)$ if $|\nu|\leq k-1$ and thus, since $\Lambda_{\alpha-k}(\omega)$ is an algebra, it remains to show that also $D^\eta H_i(z,\Phi(z))\in\Lambda_{\alpha-k}(\omega)$, $|\eta|\leq k-1$.

Put $J=D^\eta H_i$. Then $J\in\Lambda_{\alpha-k}(B)$ since $H_i\in\Lambda_{\alpha-1}(B)$. For $\alpha<k+1$ we may use first differences, and obtain immediately for z and $z+h$ belonging to ω, since $\Phi\in\mathrm{Lip}(1,\omega)$, that

$$|J(z+h,\Phi(z+h))-J(z,\Phi(z))|\leq c|(z+h,\Phi(z+h))-(z,\Phi(z))|^{\alpha-k}\leq c|h|^{\alpha-k}.$$

Suppose next that $\alpha=k+1$, and take z, $z+h$, and $z-h$ in ω. We then have, putting $G(z)=J(z,\Phi(z))$,

$$\Delta_h^2 G(z)=J(z-h,\Phi(z-h))-2J(z,\Phi(z))+J(z+h,\Phi(z)+\Phi(z)-\Phi(z-h))+J(z+h,\Phi(z+h))-$$

$J(z+h,\Phi(z)+\Phi(z)-\Phi(z-h))$ (we may assume that B contains $\omega\times 3I$, where $3I$ is the interval with same center as I but three times longer, so that the last term is defined). The sum of the first three terms constitute (we now take differences in $\mathbb{R}^n$) $\Delta_{h_1}^2 J(z,\Phi(z))$ where $h_1=(h,\Phi(z)-\Phi(z-h))$, and since $|h_1|\leq c|h|$ and $J\in\Lambda_{\alpha-k}(B)=\Lambda_1(B)$, the absolute value of this second difference is less than $c|h|$. The sum of the last two terms is $-\Delta_{h_2}^1 J(z+h,\Phi(z+h))$ where $h_2=(0,-\Delta_h^2\Phi(z))$. Since $J\in\Lambda_1(B)$, the modulus of this difference is less than $c|h_2|(1+|\ln h_2|)$ (this can e.g. be deduced with aid of the estimate on the derivatives of P_Q given after Definition 1).

Now, since $g\in\Lambda_\alpha(\mathbb{R}^n)$ where $\alpha=k+1$ we certainly have, if $k<\alpha'<k+1$, $g\in\Lambda_{\alpha'}(\mathbb{R}^n)$, and so by the non-integer case already considered, we know that at least $\Phi\in\Lambda_{\alpha'}(\overline{\omega})$. Since $\alpha'>1$ we have $\Lambda_{\alpha'}(\overline{\omega})\subset\Lambda_\gamma(\overline{\omega})$ for some γ with $1<\gamma<2$, which gives $|h_2|\leq c|h|^\gamma$ and it follows that $|\Delta_{h_2}^1 J(z+h,\Phi(z+h))|\leq c|h|$. Thus $J(z,\Phi(z))\in\Lambda_{\alpha-k}(\omega)$ also when $\alpha=k+1$, and the proposition is proved.

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