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A category analogue of the density topology, approximate continuity and the approximate derivative.

"The suggestion to look for a category analogue, or a measure analogue, has very often proved to be a useful guide." (Oxtoby, [23], p. 74)

The aim of this survey is to present a definition of a category analogue of density point and almost all results connected with this notion which were obtained in Lodz during the last three years. One can find the first, very short presentation in [36]. In what follows R will denote the real line, μ will stand for Lebesgue measure on R and N for the set of natural numbers.

In [19], [20], [21], [5], [35], and [28], p. 29 one can find the notion of category at a point. Recall that a set $E \subset R$ is said to be of the first category at a point x if there exists a neighbourhood U of x such that $U \cap E$ is of the first category. The definitions of second category at a point and (more important) of residuality at a point are formulated in an obvious way. More general notions of this kind have appeared in [13]. Although these points play some role in differentiation ([8], [21]), in cluster sets ([35]) and also in the theory of sets having the Baire property, they cannot be regarded as a very exacting analogue of dispersion and density points. When we consider first category points and residual points of a set we have only extremal possibilities: either the set is very small (first category) or it is very big (residual) in some neighbourhood of a point under consideration. It is obvious that if x is a dispersion point of E, then the measure of $E \cap U$ can be positive for each neighbourhood U of x, and if x is a density point of E, then the measure of $E \cap U$ can be smaller than the measure of U for each neighbourhood U of x. Moreover, if we define "measure zero at a point x" and "full measure at a point x" in an obvious way, then it is easy to find a measurable set E of real numbers which differs from the set of its full measure points by a set of positive measure, and this means that a Lebesgue density theorem for these points does not hold.

To obtain the category analogue of a density point which estimates the size of the set in the vicinity of a point more precisely than to say either the set is very small or very big, we shall analyze the definition of a metric) density point. Let's start with the common definition: 0 is a density point of a Lebesgue measurable set E if and only if $\lim_{h\to 0^+}[(2h)^{-1}\cdot\mu(E \cap (-h,h))] = 1$. Observe that this condition is fulfilled if and only if $\lim_{n\to\infty}[(2^{-1}n)\cdot\mu(E \cap (-1/n,1/n))] = 1$. This is equivalent to saying that $\lim_{n\to\infty} \mu((n\cdot E) \cap (-1,1)) = 2$, where $n\cdot E = \{nx\colon x \in E\}$, and this means that the sequence of characteristic functions of the sets $(nE) \cap (-1,1)$ tends in measure to the characteristic function of the interval $(-1,1)$. In the sequel the characteristic function of a set A will be denoted by χ_A. But convergence in measure can be described without measure! Indeed, if (X,S,ν) is a finite measure space, then a well-known theorem due Riesz says that a sequence $\{f_n\}_{n\in N}$ of real valued S-measurable functions converges in measure to a real valued S-measurable function f if and only if every subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges ν-a.e. to f. To describe convergence ν-a.e. one needs only the σ-ideal of sets of ν measure zero, not the measure ν itself. So from the above it follows that the notion of a density point can be described without measure. Only knowledge of the σ-ideal of null sets is necessary. Now we are ready to formulate our definition. Let S be a σ-algebra of subsets of R having the Baire property and $I \subset S$ - the σ-ideal of sets of the first category. We shall say that some property holds I-almost everywhere (in abbr. I-a.e.) if and only if the set of points which do not have this property belongs to I. We shall say also that the sequence $\{f_n\}_{n\in N}$ of S-measurable (i.e., having the Baire property) real valued functions defined on $(-1,1)$ converges with respect to I to some S-measurable real valued function f defined on $(-1,1)$ if and only if every subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges to f I-a.e. on $(-1,1)$. We shall use the denotation

$$f_n \xrightarrow[n\to\infty]{I} f.$$

Definition 1. We shall say that 0 is an $\mathcal{I}$-density pointy of a set $A \in \mathcal{S}$ if and only if

$$\chi_{((n\cdot A)\cap(-1,1))} \xrightarrow[n\to\infty]{\mathcal{I}} 1.$$

We shall say that x_0 is an $\mathcal{I}$-density point of A if and only if 0 is an $\mathcal{I}$-density point of $A - x_0 = \{x - x_0 : x \in A\}$. We shall say that x_0 is an $\mathcal{I}$-dispersion point of A if and only if x_0 is an $\mathcal{I}$-density point of $R - A$. Right-hand and left-hand $\mathcal{I}$-density and $\mathcal{I}$-dipsersion points are defined in the natural way.

Observe that 0 is an $\mathcal{I}$-dispersion point of A if and only if

$$\chi_{((n\cdot A)\cap(-1,1))} \xrightarrow[n\to\infty]{\mathcal{I}} 0.$$

Observe also that 0 is an $\mathcal{I}$-density point of A if and only if $\liminf_{n\to\infty} ((n\cdot A)\cap(-1,1))$ is residual in $(-1,1)$ and 0 is an $\mathcal{I}$-dispersion point of A if and only if $\limsup_{n\to\infty} ((n\cdot A)\cap(-1,1))$ is of the first category. These remarks are very useful in many proofs.

Looking at the definition we see that we have used only the sequence of natural numbers. This is analogous to using only intervals of the form $(-n,n)$ in the usual definition of metric density. The following theorem (See [31].) shows that all sequences tending to infinity have equal rights:

Theorem 1. The point 0 is an $\mathcal{I}$-density point of the set $A \in \mathcal{S}$ if and only if for every increasing sequence $\{t_n\}_{n\in N}$ of positive real numbers tending to infinity there exists a subsequence $\{t_{n_m}\}_{m\in N}$ such that

$$\chi_{((t_{n_m}\cdot A)\cap(-1,1))} \xrightarrow[m\to\infty]{\mathcal{I}} 1 \quad \mathcal{I}\text{-a.e.}\ .$$

The second natural question is: does the notion of $\mathcal{I}$-density point differ from the notion of residual point? The answer (positive, fortunately) is included in the following theorem (See again [31].).

Theorem 2. There exists an open set $E = \bigcup_{n=1}^{\infty}(a_n,b_n)$, where $\{b_n\}_{n\in N}$ tends decreasingly to zero, $a_{n+1} < b_{n+1} < a_n$ for each $n \in N$, such that 0 is an $\mathcal{I}$-dispersion point of E.

We shall sketch the construction to present a typical technique. Let $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ be two decreasing sequences tending to zero and such that

(i) $a_{n+1} < b_{n+1} < a_n$ for $n \in N$

(ii) $\lim\limits_{n\to\infty} \dfrac{b_n - a_n}{a_n} = 0$

(iii) $\lim\limits_{n\to\infty} \dfrac{a_n - b_{n+1}}{a_n} = 1.$

(It suffices to take for example an arbitrary $b_1 > 0$ and then $a_n = \frac{n}{n+1} b_n$ and $b_{n+1} = \frac{1}{n} a_n$). Put $E = \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$. Let $\{n_m\}_{m\in N}$ be an arbitrary increasing sequence of natural numbers. We want to show that there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that

$$X_{((n_{m_p}\cdot E)\cap(0,1))} \xrightarrow[p\to\infty]{} 0$$

I-a.e. on $(0,1)$. For each natural number m, let k_m be the smallest natural number such that $(n_m\cdot(a_{k_m}, b_{k_m})) \subset (0,1)$. The sequence $\{n_m \cdot b_{k_m}\}_{m\in N}$ is bounded, so it contains a convergent subsequence which we shall denote $\{n_{m_p}\cdot b_{k_{m_p}}\}_{p\in N}$. Let $b = \lim\limits_{p\to\infty} n_{m_p}\cdot b_{k_{m_p}}$. Two cases are possible. First case: $b = 0$. Then observe that $\liminf\limits_{p\to\infty} n_{m_p} a_{k_{m_p}-1} \geq 1$. Indeed, if $\liminf\limits_{p\to\infty} n_{m_p} a_{k_{m_p}-1} < 1$, then from (ii) it follows that for infinitely many p's we should have $n_{m_p}\cdot b_{k_{m_p}-1} < 1$. This is a contradiction to the definition of k_m. So

$$\{X_{((n_{m_p}\cdot E)\cap(0,1))}\}_{p\in N}$$

converges to 0 at every point of $(0,1)$. Second case: $b > 0$. Then observe that $\lim_{p\to\infty} n_{m_p} \cdot a_{k_{m_p}-1} = +\infty$. (This follows from the fact that $\lim_{n\to\infty} \frac{a_{n-1}}{b_n} = +\infty$, which is a consequence of (iii).) Also $\lim_{p\to\infty} n_{m_p} \cdot a_{k_{m_p}} = b$, because from (ii) we have $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\lim_{p\to\infty} n_{m_p} \cdot b_{k_{m_p}+1} = 0$. (This is again a consequence of (iii).) Hence $\limsup_{p\to\infty} ((n_{m_p} \cdot E) \cap (0,1)) \subset \{b\}$, and because a singleton $\{b\}$ is of the first category, we conclude that 0 is an I-dispersion point of E.

Observe that sometimes it is more difficult to prove a fact about I-density than to prove the corresponding fact about metric density because in the process of choosing subsequences there is no possibility of estimating the size of sets under consideration. However, there are also conditions under which the situation is simpler: namely, each set having the Baire property differs from an open (or closed) set by a set of the first category (while a Lebesgue measurable set in general can be approximated up to null set by F_σ or G_δ set). Consequently numerous theorems in the sequel shall be formulated for open (or closed) sets having the Baire property.

Having this last remark in mind and using the notation $\Phi(A) = \{x \in R$: x is an I-density point of $A\}$ for $A \in S$; while $A \sim B$ means that $A \triangle B \in I$ we obtain the following theorem:

<u>Theorem 3</u>. For every $A,B \in S$

1) $\Phi(A) \sim A$

2) if $A \sim B$, then $\Phi(A) = \Phi(B)$

3) $\Phi(\emptyset) = \emptyset$, $\Phi(R) = R$

4) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

The above theorem has been proved in [30]. From this we see that the operator Φ transforms S into S and is the so-called "lower density". (See [28], Th. 22.4.) Put $T_I = \{\Phi(A) - I: A \in S, \; I \in I\}$. Similar to Th. 22.5 in [28] (using the fact that every disjoint family of sets of the second category having the Baire property is at most enumerable) we can prove:

Theorem 4. ([30]). T_I is a topology on the real line.

The only difficult part is to prove that the arbitrary union of sets in T_I belongs to S. Observe that $T_I = \{A \in S: A \subset \Phi(A)\}$; this means that T_I consists of all sets in S with the property that each point of the set is an I-density point of the set. This description is more familiar (also for the density topology - see [9] and [10]). Obviously T_I is stronger than the natural topology on the real line, so it is a Hausdorff topology. But if Q is the set of rational numbers, then $\{0\}$ and $Q - \{0\}$ cannot be separated by sets from T_I, and hence T_I is not a regular topology. Here are some other properties of T_I (See [30] and [31].): the topological space (R, T_I) is not a separable space and any closed interval is connected but not compact in this space.

It is not clear at first glance if there is some connection between density and I-density points or dispersion and I-dispersion points. The following theorem (See [1].) explains that there is no connection. Observe that c) and d) are not simple consequences of taking complements in a) and b) and the construction of the example in d) is rather complicated.

Theorem 5. a) There exists an open set E_1 such that 0 is a density point of E_1 but not an I-density point of E_1.

b) There exists an open set E_2 such that 0 is an I-density point of E_2 but not a density point of E_2.

c) There exists an open set E_3 such that 0 is a dispersion point of E_3 but not an I-dispersion point of E_3.

d) There exists an open set E_4 such that 0 is an I-dispersion point of E_4 but not a dispersion point of E_4.

Let us introduce the following definition.

Definition 2. We shall say that a set $A \subset R$ is a T_I-neighbourhood of $x \in R$ if and only if there exists a set $A_1 \in S$ such that $A_1 \subset A$ and x is an I-density point of A_1. (Compare to [29].)

We have the following theorem which corresponds to the well-known theorem about the density topology:

Theorem 6. ([29]) A set $A \subset R$ belongs to T_I if and only if A is a T_I-neighbourhood of each of its points.

Observe that the non-trivial part of this theorem says that A has the Baire property which is not supposed in the definition of T_I-neighbourhood.

In the sequel we shall denote by $\mathrm{Int}_{T_I} A$, $\mathrm{Cl}_{T_I} A$, $\mathrm{Fr}_{T_I} A$ and A^{dT_I} the interior, closure, boundary and derivative, respectively, of the set A in the topology T_I. The meaning of T_I-open, T_I-closed set is obvious. The same symbols without index T_I are always used for the natural topology on the real line. The following theorems show that the properties of these operations are very similar to those of the density topology.

Theorem 7. ([29]). If $A \in S$, then $\mathrm{Int}_{T_I} A = A \cap \Phi(A)$, $\mathrm{Cl}_{T_I} A = A \cup (R - \Phi(R - A))$, $\mathrm{Fr}_{T_I} A \in I$.

From the last part it follows immediately that $\mathrm{Cl}_{T_I} A - A \in I$ and $A - \mathrm{Int}_{T_I} A \in I$. Observe that if $A \notin S$, then it can happen that $\mathrm{Cl}_{T_I} A - \mathrm{Int}_{T_I} A \notin I$.

Theorem 8. ([29]). If $A, B \in S$, then $A^{dT_I} = B^{dT_I}$ if and only if $A \sim B$. If $A \in S$, then $A^{dT_I} = R - \Phi(R - A)$.

If we define the operation $\kappa \colon 2^R \to S$ in the following way: $\kappa(A) = \Phi(\mathrm{Cl}_{T_I} A)$, then we have:

Theorem 9. ([29]). If $A \in S$, then $\kappa(A) = \Phi(A)$.

Theorem 10. ([29]). If $A^{dT_I} = R - \kappa(R - A)$, then $A \in S$. (Compare also to [39].)

The following results were obtained (among others) in [29] for the operator $\Delta: 2^R \to S$ defined (in the same fashion as the similar operator in [24] for the density topology) by the formula $\Delta(A) = Cl_{T_I}(Int_{T_I} A)$.

Theorem 11. $A - \Delta(A) \subset Fr_{T_I} A$. If $A_1 \subset A_2$, then $\Delta(A_1) \subset \Delta(A_2)$. $\Delta^2(A) = \Delta(\Delta(A)) = \Delta(A)$.

Theorem 12. If $A \in S$, then $A - \Delta(A) \in I$; $\Delta(A) = \emptyset$ if and only if $A \in I$; $Cl_{T_I} A = A \cup \Delta(A)$.

Now we shall study real functions of a real variable. (See [30] and [31].)

Definition 3. We shall say that a function $f: R \to R$ is I-approximately continuous at x_0 if and only if for every $\epsilon > 0$ the set $f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$ has x_0 as an I-density point. We shall say that a function $f: R \to R$ is I-approximately continuous if and only if for every interval (y_1, y_2) the set $f^{-1}((y_1, y_2))$ belongs to T_I. (This means simply that f is a continuous transformation from (R, T_I) into R equipped with the natural topology.)

Recall that in real analysis there are at least two frequently used definitions of approximate continuity at a point x_0. The first of them (similar to Definition 3 above) says that f is approximately continuous at x_0 if and only if for every $\epsilon > 0$ the set $f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$ has x_0 as a density point, the second deals with some restriction of f, namely, f is approximately continuous at x_0 if and only if there exists a neighbourhood E of x in the density topology such that $f|_E$ is continuous at x_0 (in the natural topology restricted to E). If we should take any topology T instead of the density topology, we would obtain the "topological" and "restricted" definitions of continuity at x. According to [14] (Theorem 5) these conditions are equivalent for a topology T invariant with respect to translations if and only if the following condition is fulfilled (We quote this condition in the formulation convenient for our purposes.): for every descending sequence $\{E_n\}_{n\in N}$ of right-hand (left-hand) T-neighbourhoods of 0 there exists a sequence $\{h_n\}_{n\in N}$ such that $h_n > 0$, $\lim_{n\to\infty} h_n = 0$, and the set

$$\{0\} \cup \bigcup_{n=1}^{\infty} (E_n \cap [h_{n+1},h_n)) \quad \cup \quad (\{0\} \cup \bigcup_{n=1}^{\infty} (E_n \cap (-h_n,-h_{n+1}]))$$

includes a right-hand (left-hand) T-neighbourhood of 0.

Since for any descending sequence $\{E_n\}_{n\in N}$ of sets having 0 as a point of right-hand density there exists a sequence $\{h_n\}_{n\in N}$, $h_n > 0$, $\lim_{n\to\infty} h_n = 0$ for which the set $\bigcup_{n=1}^{\infty} (E_n \cap [h_{n+1},h_n))$ also has 0 as a point of right-hand density, the above quoted two definitions of approximate continuity are obviously equivalent. Observe also that the above condition can be formulated in terms of points of dispersion.

From the following theorem we can conclude immediately that for the topology T_I the "topological" and "restricted" definitions are not equivalent. Obviously "restricted" continuity always implies "topological".

<u>Theorem 13</u>. ([31]). There exists an increasing sequence $\{A_n\}_{n\in N}$ of sets having the Baire property such that for every natural n, 0 is an I-dispersion point of A_n and for any sequence $\{h_n\}_{n\in N}$ of numbers tending decreasingly to zero, the point 0 is not an I-dispersion point of the set

$$A = \bigcup_{n=1}^{\infty} (A_n \cap [h_{n+1},h_n)).$$

The construction is based upon two easy observations: If 0 is an I-dispersion point of B_1 and B_2, then 0 is an I-dispersion point of $B_1 \cup B_2$. If 0 is an I-dispersion point of B, then for every $a \in R$, 0 is an I-dispersion point of $a\cdot B$. Let E be the set constructed in

Theorem 2. Put $A_1 = E$ and $A_n = A_{n-1} \cup \bigcup_{i=1}^{\infty} (\frac{i}{n} \cdot E)$ for $n > 1$. The sequence $\{A_n\}_{n\in N}$ fulfills all requirements.

In the sequel we shall not pay attention to "restricted" continuity, taking into account only the "topological" version.

<u>Theorem 14</u>. ([31]). A function $f: R \to R$ has the Baire property if and only if it is I-approximately continuous I-a.e..

The above theorem as well as the next one have very well-known analogues for Lebesgue measurable function.

<u>Theorem 15</u>. ([30]). If a function $f: R \to R$ is I-approximately continuous, then f is in the first Baire class and has the Darboux property.

In the mathematical literature one can find several proofs of similar theorems for approximate continuity. However, most of them use some properties of the Lebesgue integral. Our proof is in a way similar to the proof in [9] (which does not use the integral). A very essential role is played by the following lemma.

<u>Lemma 1</u>. ([30]). If 0 is an I-density point of the set $A \in S$, then for every natural number n there exists a number $\delta_n > 0$ such that for each h fulfilling the inequality of $0 < h < \delta_n$ and for every natural number k such that $-n \leq k \leq n-1$ we have $A \cap \{[\frac{k}{n} \cdot h, \frac{k+1}{n} \cdot h] \neq \emptyset$.

The proof that f is in the first Baire class is difficult. The Darboux property follows immediately from it and Theorem II.1.1. in [4].

The connectedness of $[a,b]$ in T_I (mentioned earlier) is an easy consequence of Theorem 15.

Observe also that in [1] one can find an example of an I-approximately continuous function having a (specially constructed) perfect nowhere dense set as the set of points of discontinuity (with respect to the natural topology). The construction is straightforward.

If we denote by A and A_I the class of approximately and

I-approximately continuous functions, respectively, then using Theorem 5 we immediately obtain the following result.

Theorem 16. Neither $A_I \subset A$, nor $A \subset A_I$.

We can also compare the class A_I with some classes of derivatives. We use the following notation: Δ (derivatives), bΔ (bounded derivatives), bA_I (bounded I-approximately continuous functions) and obtain the result which differs slightly from Theorem II.5.5. in [4]: b$A_I \not\subset$ bΔ, $A_I \not\subset \Delta$, b$\Delta \not\subset$ bA_I.

It is known that the density topology is completely regular ([9]), so it is the coarsest topology relatively to which all approximately continuous functions are continuous. As we observed earlier, for the topology T_I the situation is different - is is not regular. Already in [30] it was proved that if f is an I-approximately continuous function, then for each open interval (y_1,y_2) the set $f^{-1}((y_1,y_2))$ is of the form $G \cup Z$, where G is open (in the natural topology) and Z is of the first category. If $T^1{}_I = \{U \in T_I: U = G \cup Z$, where G is open and Z is of the first category$\}$, then it is not difficult to verify (See [29].) that $T^1{}_I$ is a topology stronger than a natural topology but weaker than T_I. Also it is easy to observe that the topological space $(R,T^1{}_I)$ is connected and that $T^1{}_I$-compact sets are merely the finite sets.

It is natural to seek the coarsest such topology corresponding to the I-approximately continuous functions. It was found independently in [17] and [29]. It turned out that $T^1{}_I$ is still too strong. Now we shall present this coarsest topology starting from some facts proved in [29]. Denote by $C(R,V)$ the family of all real functions which are continuous with respect to some topology V. Then $C(R,T^1{}_I) = C(R,T_I) = A_I$. We shall introduce two definitions of some variations of I-density points.

Definition 4. A point x_0 is called a special (deep) I-density point of $A \in S$ if and only if there exists an open set C such that $\mathrm{Cl}\, C \supset R - A$ ($C \supset R - A$) and x_0 is an I-dispersion point of C.

The first part of the definition is justified by the following theorem:

__Theorem 17__. If a function $f:R \to R$ is I-approximately continuous, then for every interval (y_1,y_2) each point x_0 of the set $A = f^{-1}(y_1,y_2))$ is a special I-density point of A.

Looking at Definition 4 one can observe without difficulty that deep I-density points are much more convenient to handle. At this point we are fortunate because the following theorem holds:

__Theorem 18__. If $A \in T^1{}_I$, then x_0 is a special I-density point of A if and only if it is a deep I-density point of A.

Observe also that each point which is a deep I-density point of A belongs to A.

Now let $T = \{A \in T^1{}_I$: every point $x \in A$ is a deep I-density point of $A\}$. It is easy to see that T is a topology on the real line between the natural topology and $T^1{}_I$. Observe that the set

$$A = R - \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \{\frac{k}{(n+1)!}\}$$

belongs to $T^1{}_I$, but not to T, because 0 is not a deep I-density point of A. For T the analogue of Lusin-Menchoff Theorem is valid.

__Theorem 19__. Let $E \in T^1{}_I$ and let $F \subset E$ be a closed set such that every point $x \in F$ is a deep I-density point of E. Then there exists a perfect set P such that $F \subset P \subset E$ and every point $x \in F$ is a deep I-density point of P.

From this theorem it is not difficult to deduce that (R,T) is a completely regular (Tychonoff) topological space. By virtue of the fact that $C(R,T_I) = C(R,T)$ (This is a consequence of Theorems 17 and 18.) we obtain:

__Theorem 20__. T is the coarsest topology for which every I-approximately continuous function is continuous.

In [17] there is the following description of the topology T:

Let $P(x)$ denote the collection of all closed intervals $[a,b]$ such that $x \in (a,b)$ and of all sets of the form $\bigcup_{n=1}^{\infty} [a_n,b_n] \cup \bigcup_{n=1}^{\infty} [c_n,d_n] \cup \cup \{x\}$ or $\bigcup_{n=1}^{\infty} [a_n,b_n] \cup [c_1,x]$ or $\bigcup_{n=1}^{\infty} [c_n,d_n] \cup [x,b_1]$, where for all n $x < b_{n+1} < a_n < b_n$ and $c_n < d_n < c_{n+1} < x$ and x is the limit of $\{a_n\}_{n\in N}$ and of $\{c_n\}_{n\in N}$.

Theorem 21. ([17]). Let T be the collection of all sets $U \in T_I$ fulfilling the following condition: if $U \neq \emptyset$, then for every $x \in U$ there exists a set $P \in P(x)$ such that $P \subset \text{Int } U \cup \{x\}$ and x is an I-density point of P. Then T is the coarsest topology for which all I-approximately continuous functions are continuous.

We used the same letter T here as earlier for the obvious reason.

Let us remark that in both papers [29] and [17] there are some other interesting observations concerning the topologies $T^1{}_I$ and T.

Recall that a well-known theorem of Maximoff ([4], p. 36 or [32]) says that if $f:R \to R$ is Darboux and Baire one, then there exists a homeomorphism h of R onto itself such that $f \circ h$ is approximately continuous. In [1] it was observed that there exists a function $f:R \to R$ which is Darboux and Baire one and for every homeomorphism h of R onto itself $f \circ h \notin A_I$. Any function f which is Darboux and Baire one and $f = 0$ a.e. but not everywhere has this property. (Ccompare to [4], p. 12.) Using Theorem 21 we can observe that if a function $f:R \to R$ can be transformed by an inner homeomorphism onto an I-approximately continuous function, then for each interval (y_1,y_2) and each point $x_0 \in f^{-1}((y_1,y_2))$ there exists a set $P \in P(x)$ such that $P \subset \text{Int } f^{-1}((y_1,y_2)) \cup \{x_0\}$. It is natural to ask if this condition (it is a kind of two-sides quasi-continuity in the sense of Kempisty) together with Darboux and Baire one implies the existence of a homeomorphism transforming a given function into an I-approximately continuous function. Unfortunately, the answer is not known as yet. We know only the following theorem for sets: (Compare to [11].)

Theorem 22. ([33]). If $U \subset R$ is a set having the following property: for each $x \in U$ there exists a set $P \in \mathcal{P}(x)$ such that $P \subset \operatorname{Int} U \cup \{x\}$, then there exists a homeomorphism h of R onto itself such that $h(U) \subset \subset \Phi(h(U))$.

After this short and incomplete information about creating $\mathcal{I}$-approximate continuity we shall discuss some results on preserving $\mathcal{I}$-density and $\mathcal{I}$-approximate continuity as they were presented in [2]. Recall that homeomorphisms preserving density points were studied by Bruckner (See [4], p. 138.) and Niewiarowski in [23]. We shall consider only increasing homeomorphisms $h:R \to R$.

Definition 5. We say that a homeomorphism h preserves $\mathcal{I}$-density (deep $\mathcal{I}$-density) at x_0 if and only if for each set $A \in \mathcal{S}$ if x_0 is an $\mathcal{I}$-density (deep $\mathcal{I}$-density) point of A, then $h(x_0)$ is an $\mathcal{I}$-density (deep $\mathcal{I}$-density) point of $h(\bar{A})$. We say that h preserves $\mathcal{I}$-density (deep $\mathcal{I}$-density) if it preserves $\mathcal{I}$-density (deep $\mathcal{I}$-density) at each point $x_0 \in R$.

It is easy to see that in the above definition we can use only open sets instead of sets having the Baire property. The following theorems are somewhat more difficult to prove:

Theorem 23. A homeomorphism preserves $\mathcal{I}$-density if and only if it preserves deep $\mathcal{I}$-density.

Theorem 24. A homeomorphism h preserves $\mathcal{I}$-density at x_0 if and only if it preserves $\mathcal{I}$-density for sets belonging to $\mathcal{P}(x_0)$.

The last theorem allows us to simplify many of the proofs in [2]. Using it we can immediately obtain the following:

Theorem 25. If h_1 and h_2 preserve $\mathcal{I}$-density, then $h_1 + h_2$, $\max(h_1,h_2)$ and $\min(h_1,h_2)$ also preserve $\mathcal{I}$-density.

Theorem 26. If h and h^{-1} fulfill a local Lipschitz condition, then h preserves I-density.

As with the analogous result concerning density preservation, the above-mentioned condition is not necessary. It is not difficult to give examples (See again [2].) which preserve I-density without fulfilling a Lipschitz condition (neither by h, nor by h^{-1}). Also there exists a homeomorphism h preserving I-density such that h^{-1} does not preserve I-density. The following theorem shows the similarity between approximate continuity and I-approximate continuity (Compare to [4], p. 138.).

Theorem 27. A function $f \circ h$ is I-approximately continuous for every I-approximately continuous function f if and only if $g = h^{-1}$ preserves I-density.

We shall conclude the part of a paper devoted to I-approximate continuity of functions of one variable with two problems. Neither of them has a complete solution as yet. The first is connected with the characterization of the set of points of I-approximate continuity and was studied in [22] where one can find the following theorem.

Theorem 28. If $B = \bigcap_{n=1}^{\infty} G_n \cup J$, where G_n is open (in the natural topology) for each $n \in N$, $J \in I$, $J \cap \bigcap_{n=1}^{\infty} G_n = \emptyset$ and $J \subset \bigcap_{n=1}^{\infty} \Phi(G_n)$, and if $\bigcap_{n=1}^{\infty} \Phi(G_n) - B$ is countable, then there exists a real function f of a real variable for which B is exactly the set of all points of I-approximate continuity.

Observe that if $R - B \notin I$, then by virtue of Theorem 14, f cannot have the Baire property. It is unknown whether in the above theorem the countability of $\bigcap_{n=1}^{\infty} \Phi(G_n) - B$ is necessary.

The second problem is connected with the characterization of the class $B_1(A_I)$ of pointwise limits of sequences of I-approximately continuous functions. The following theorem can be found in [3]. (Compare to [12].)

Theorem 29. Let $f \in B_1(A_I)$. Then the following condition holds: for any $a,b \in R$, $a < b$ and nonempty sets U,V if

(1) $U \subset \{x: f(x) < a\}$

(2) $V \subset \{x: f(x) > b\}$

(3) $U \subset \Delta(\mathrm{Cl}\ U)$ and $V \subset \Delta(\mathrm{Cl}\ V)$, then

$U - \mathrm{Cl}\ V \neq \emptyset$ and $V - \mathrm{Cl}\ U \neq \emptyset$.

Observe that the condition $U \subset \Delta(\mathrm{Cl}\ U)$ means that the closure of U in the topology T is a perfect set in this topology. (See [17] and [29].) Using the above theorem and some special sets belonging to the families $P(x)$ the authors in [3] have proved the following theorem.

Theorem 30. $B_1 \subsetneq B_1(A_I) \subsetneq B_2$.

Here B_1 and B_2 denote the first and the second Baire classes for the natural topology.

The problem of finding a complete characterization of $B_1(A_I)$ remains unsolved.

Now let us pay some attention to I-density points of plane sets. (See [6].) Here S and I will denote the classes of subsets of the plane having the Baire property and the σ-ideal of sets of the first category.

Definition 6. A point $(0,0)$ is an I-density point of $A \in S$ if and only if for every two increasing sequences $\{k_n'\}_{n\in N}$, $\{k_n''\}_{n\in N}$ of natural numbers for which there exists a number $\alpha > 1$ such that for each $n \in N$

$\frac{1}{\alpha} < \frac{k_n'}{k_n''} < \alpha$ the following holds:

$\{\chi_{((k_n',k_n'')\cdot A) \cap ([-1,1] \times [-1,1])}\}_{n\in N}$ converges to 1 with respect to I.

A point $(0,0)$ is a strong I-density point of A if and only if for every two increasing sequences $\{k_n'\}_{n\in N}$, $\{k_n''\}_{n\in N}$ of natural numbers

$\{\chi_{((k_n',k_n'')\cdot A) \cap ([-1,1] \times [-1,1])}\}_{n\in N}$ converges to 1 with respect to I.

The definitions for (x_o, y_o) in place of $(0,0)$ are obtained by translation as in the one dimensional case.

Obviously in the above definition $(a,b)\cdot A = \{(ax,by): (x,y) \in A\}$. It is not difficult to prove that in the definition of I-density point one can use the sequence (k_n, k_n) instead of (k_n', k_n''). Of course each strong I-density point is an I-density point of A, but not conversely. If for $A \in S$ we denote by $\Phi(A)$ the set of all I-density points of A and by $\Phi_S(A)$ the set of all strong I-density points of A, then the operators Φ and Φ_S have the same properties as the one-dimensional operator Φ (Theorem 3). Exactly in the same way we define topologies T_I and $T^S{}_I$ using Φ and Φ_S, respectively. These topologies have the same properties as the one dimensional topology T_I. I-approximately continuous functions of two variables (and strongly I-approximately continuous functions, which constitute a smaller class) are Baire one. The coarsest topology for which all I-approximately continuous functions are continuous is described exactly as in the one dimensional case using the notion of deep I-density. The problem of finding the coarsest topology for strongly I-approximately continuous functions is open. However, there are some partial results in this direction connected with the notion of deep strong I-density. Also the following theorem is true. (Compare to [9].)

Theorem 31. If $f:R^2 - R$ is strongly I-approximately continuous, then for each $x_o \in R$, $f(x_o, y)$ is I-approximately continuous as a funtion of y.

Using a method similar to that in [7] the following theorem has been proved.

Theorem 32. ([37]). If the function $f:R^2 - R$ has the following property: if for each $x_o \in R$, $f(x_o, y)$ is I-approximately continuous as a function of y and for each $y_o \in R$, $f(x, y_o)$ is I-approximately continuous as a function of x, then f has the Baire property (as a function of two variables).

It is an open problem as to whether the function from Theorem 32 is Baire one. The method of proof in [7] uses the integral and, as a consequence, cannot be adopted here.

In the next part of the paper we shall present some results concerning the category analogue of the approximate derivative. We shall start with a definition. (Compare to [34].)

Definition 7. ([18]). A collection $\mathcal{C}$ of closed subintervals of an interval $[a,b]$ is said to be an $\mathcal{I}$-approximate full cover of $[a,b]$ if and only if to each $x \in [a,b]$ there corresponds a set $A_x \in \mathcal{S}$ such that x is an $\mathcal{I}$-density point of A_x (right-hand $\mathcal{I}$-density point if $x = a$, left-hand $\mathcal{I}$-density point if $x = b$) and every interval I with $x \in I$ and with endpoints in A_x belongs to $\mathcal{C}$.

The following lemma is also in [18].

Lemma 2. If 0 is an $\mathcal{I}$-dispersion point of the open set G, then there exists a natural number k and a real number $\delta > 0$ such that for every $h \in (0,\delta)$ there exist two natural numbers $i_r, i_t \in \{1,\dots,k\}$ such that

$$\left(\frac{i_r-1}{k}\cdot h, \frac{i_r}{k}\cdot h\right) \cap G = \emptyset \quad \text{and} \quad \left(\frac{-i_t}{k}\cdot h, \frac{-i_t+1}{k}\cdot h\right) \cap G = \emptyset.$$

This lemma together with Lemma 1 enables us to represent the interval $[a,b]$ as the countable union $[a,b] = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} E_{k,j}$ with the property:

if $x_1,x_2 \in E_{k,j}$ and $0 < x_2 - x_1 < \frac{1}{j}$, then $A_{x_1} \cap A_{x_2} \cap [x_1,x_2] \neq \emptyset$

(where A_{x_1} and A_{x_2} are sets associated with x_1 and x_2 in the above definition). This decomposition is the main tool in the proof of the following theorem.

Theorem 33. ([18]). If $\mathcal{C}$ is an $\mathcal{I}$-approximate full cover of an interval $[a,b]$, then $\mathcal{C}$ contains a partition of every subinterval of $[a,b]$.

By a partition of an interval we mean a finite family of non-overlapping intervals the union of which is the given interval. In the proof of the corresponding theorem in [34] the following fact has been used: there exists a decomposition $[a,b] = \bigcup_{n=1}^{\infty} E_n$ with the property that if x_1, $x_2 \in E$, $0 < x_2 - x_1 < \frac{1}{n}$, then $A_{x_1} \cap A_{x_2} \cap [a,b] \neq \emptyset$, where A_{x_1} and A_{x_2} are the sets associated with x_1 and x_2 in the definition of Thomson (x_i is a density point of A_i, $i = 1,2$). To obtain this decomposition it suffices to take $E_n = \{x:$ for each $h \in (0,\frac{1}{n}]$, $h^{-1} \cdot \mu(A_x \cap (x - h, x + h)) > \frac{1}{2}\}$. The construction of the decomposition in our case is more complicated.

Having the above theorem at our disposal we are able to repeat the proofs of all theorems from [34] for the case of the I-approximate derivative. However, in the sequel we shall prsent some stronger theorems. Now we come to the definition of I-approximate derivative.

Definition 8. ([18]). Suppose $f:R \to R$ has the Baire property in a neighbourhood of x_0. The upper I-approximate limit of f at x_0 ($\limsup_{x\to x_0} I\text{-ap}\, f(x)$) is the greatest lower bound of the set $\{y: \{x: f(x) > y\}$ has x_0 as an I-dispersion point$\}$. The lower I-approximate limit ($\liminf_{x\to x_0} I\text{-ap}\, f(x)$) right-hand and left-hand upper and lower I-approximate limits etc. are defined similarly. If $\limsup_{x\to x_0} I\text{-ap}\, f(x) = \liminf_{x\to x_0} I\text{-ap}\, f(x)$, their common value is called the I-approximate limit of f at x_0 and denoted by $\lim_{x\to x_0} I\text{-ap}\, f(x)$.

Definition 9. ([18]). Let $f:R \to R$ be any function with the Baire property in some neighbourhood of x_0, and let $C(x,x_0) = (f(x)-f(x_0))/(x-x_0)$ for $x \neq x_0$. We define the I-approximate upper right derivate ($D^+{}_{I\text{-ap}}f(x_0)$); the I-approximate lower right derivate ($D_+{}_{I\text{-ap}}f(x_0)$); the I-approximate upper left derivate ($D^-{}_{I\text{-ap}}f(x_0)$); the I-approximate lower left derivate ($D_-{}_{I\text{-ap}}f(x_0)$); the I-approximate bilateral upper derivate $\bar{f}'_{I\text{-ap}}(x_0)$); the I-approximate bilateral lower derivate ($\underline{f}'_{I\text{-ap}}(x_0)$) as the corresponding extreme limits of $C(x,x_0)$ as x tends to x_0. When all of these derivates are equal and finite we call their common value the $\mathcal{I}$-approximate derivative of f at x_o and denote it by $f'_{\mathcal{I}\text{-ap}}(x_o)$.

In [25] one can find a theorem saying that the approximate derivative (if it exists everywhere) is a selective derivative for some selection. In [18] we proved that for the $\mathcal{I}$-approximate derivative the situation is similar.

Theorem 34. If $f:[0,1] \to R$ has a finite $\mathcal{I}$-approximate derivative $f'_{\mathcal{I}\text{-ap}}(x)$ at all $x \in [0,1]$, then there exists a selection S such that $sf'(x) = f'_{\mathcal{I}\text{-ap}}(x)$ for all x.

This theorem at once becomes a powerful tool for proving several theorems. (Compare to [25], [26], [15] and [4], pp. 155-157.)

Theorem 35. Suppose $f: [0,1] \to R$ has a finite $\mathcal{I}$-approximate derivative $f'_{\mathcal{I}\text{-ap}}(x)$ for all $x \in [0,1]$. Then

(a) There is a sequence of closed sets $\{E_n\}_{n\in N}$ whose union is $[0,1]$ such that f is continuous on each E_n relatively to E_n.

(b) The function f has the Darboux property.

(c) There is a dense open set U on which f is continuous.

(d) The $\mathcal{I}$-approximate derivative has the Darboux property.

(e) The set of points of continuity of $f'_{\mathcal{I}\text{-ap}}(x)$ is dense in $[0,1]$ and f is defferentiable at any point of continuity of the $\mathcal{I}$-approximate derivative.

(f) f is differentiable for $\mathcal{I}$-almost all x in $[0,1]$.

(g) The function f has an approximate derivative $f'_{ap}(x)$ for almost all $x \in [0,1]$.

Theorem 36. ([18]). Suppose $f:[0,1] \to R$ has an $\mathcal{I}$-approximate derivative at all $x \in [0,1]$. If $a = \inf_x (f'_{\mathcal{I}\text{-ap}}(x))$,

$$b = \inf_{x\neq y} \frac{f(x)-f(y)}{x-y}, \qquad c = \sup_x (f'_{\mathcal{I}\text{-ap}}(x)), \qquad d = \sup_{x\neq y} \frac{f(x)-f(y)}{x-y},$$

then $a = b$ and $c = d$.

Theorem 37. ([18]). Let f be an increasing function defined on $[0,1]$. Then, for each $x \in (0,1)$, $D_+f(x_0) = D_{+\,\mathcal{I}\text{-ap}}f(x_0)$. The corresponding equalities for the other extremal derivates and extremal $\mathcal{I}$-approximate derivates are also valid.

Theorem 38. ([18]). Let f be increasing on an interval $[0,1]$. If f is $\mathcal{I}$-approximately differentiable at $x_0 \in (0,1)$, then f is differentiable at x and $f'(x_0) = f'_{\mathcal{I}\text{-ap}}(x_0)$.

Theorem 39. ([18]). If f is $\mathcal{I}$-approximately differentiable on $[0,1]$ and $f'_{\mathcal{I}\text{-ap}}(x) > 0$ for all $x \in [0,1]$, then f is nondecreasing on $[0,1]$.

Theorem 40. ([18]). Let f be $\mathcal{I}$-approximately differentiable on $[0,1]$ and let g be differentiable on $[0,1]$. If $f'_{\mathcal{I}\text{-ap}}(x) \leqslant g'(x)$ for all $x \in [0,1]$, then f is differentiable on $[0,1]$ and $f'(x) = f'_{\mathcal{I}\text{-ap}}(x)$ for all x.

Theorem 41. ([18]). (Mean Value Theorem). If f is $\mathcal{I}$-approximately differentiable on $[0,1]$, then to each pair of numbers $a,b \in [0,1]$ there corresponds a number c between a and b such that

$$\frac{f(b)-f(a)}{b-a} = f'_{\mathcal{I}\text{-ap}}(c).$$

Theorem 42. ([18]). Suppose $f:[0,1] \to R$ has a finite $\mathcal{I}$-approximate derivative at all $x \in [0,1]$ and $a,b \in R$, $a < b$. If $A = \{x\colon a < f'_{\mathcal{I}\text{-ap}}(x) < b\} \neq \emptyset$, then the Lebesgue measure of A is positive.

The following general theorem on monotonicity (Compare to [4], p. 181.) is also in [18].

Theorem 43. Let $f:[0,1] \to R$ be Darboux and Baire one. Assume that

(i) $f'_{\mathcal{I}\text{-ap}}(x)$ exists except on a denumerable set

(ii) $f'_{\mathcal{I}\text{-ap}}(x) \geqslant 0$ almost everywhere.

Then f is non-decreasing and continuous on $[0,1]$.

From the fact that the $\mathcal{I}$-approximate derivative is equal to a selective derivative it follows immediately that the I-approximate derivative is in the second Baire class. (See [15].) Recently it was shown that this result concerning the Baire class of the I-approximate derivative can be strengthened. Mrs. Lazarow has proved the following characterization of I-dispersion points ([16]).

Theorem 44. A point 0 is the dispersion point of the open set G if and only if for every natural number n there exists a natural number k and a real number $\delta > 0$ such that for each $h \in (0,\delta)$ and for each $i \in \{1,\ldots,n\}$ there exist two natural numbers $j_r, j_t \in \{1,\ldots,k\}$ such that

$$G \cap \left[\frac{(i-1)\cdot k + j_r - 1}{n\cdot k}\cdot h,\ \frac{(i-1)\cdot k + j_r}{n\cdot k}\cdot h\right] = \emptyset \qquad \text{and}$$

$$G \cap \left[-\frac{(i-1)\cdot k + j_t}{n\cdot k}\cdot h,\ -\frac{(i-1)\cdot k + j_t - 1}{n\cdot k}\cdot h\right] = \emptyset.$$

Using this characterization she was able to prove that in Theorem 34 one can choose a balanced selection. From this fact the next theorem follows immediately (see [27]):

Theorem 45. ([16]). Suppose $f:[0,1] \to R$ has a finite I-approximate derivative $f'_{I\text{-ap}}(x)$ at all $x \in [0,1]$. Then $f'_{I\text{-ap}}(x)$ is of Baire class one.

Recall once more that we still assumed that all sets under considerations have the Baire property or are Lebesgue measurable (with only one exception in Theorem 6). This is essential because if we consider the following conditions:

1. 0 is an I-density point of some set $B \subset A$ having the Baire property.
2. $\chi_{((n\cdot A)\cap[-1,1])} \xrightarrow[n\to\infty]{I} 1.$
3. For each increasing sequence $\{t_n\}_{n\in N}$ of positive real numbers tending to infinity there exists a subsequence $\{t_{n_m}\}_{m\in N}$ such that $\chi_{((t_{n_m}\cdot A)\cap[-1,1])} \xrightarrow[n\to\infty]{} 1$ I-a.e.,

then for $A \in S$ all three conditions are equivalent. However, if $A \notin S$, then it can happen that each of them has a different meaning. (See [38].) (Observe that it is possible that a sequence of functions without the Baire property converges I-a.e. to a function having the Baire property). In the case of density points (in which case we must make natural changes in all above conditions) the situation is quite similar. Obviously we can define a topology for each kind of density (or I-density) described above, but the studies of properties of these topologies are in statu nascsendi.

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